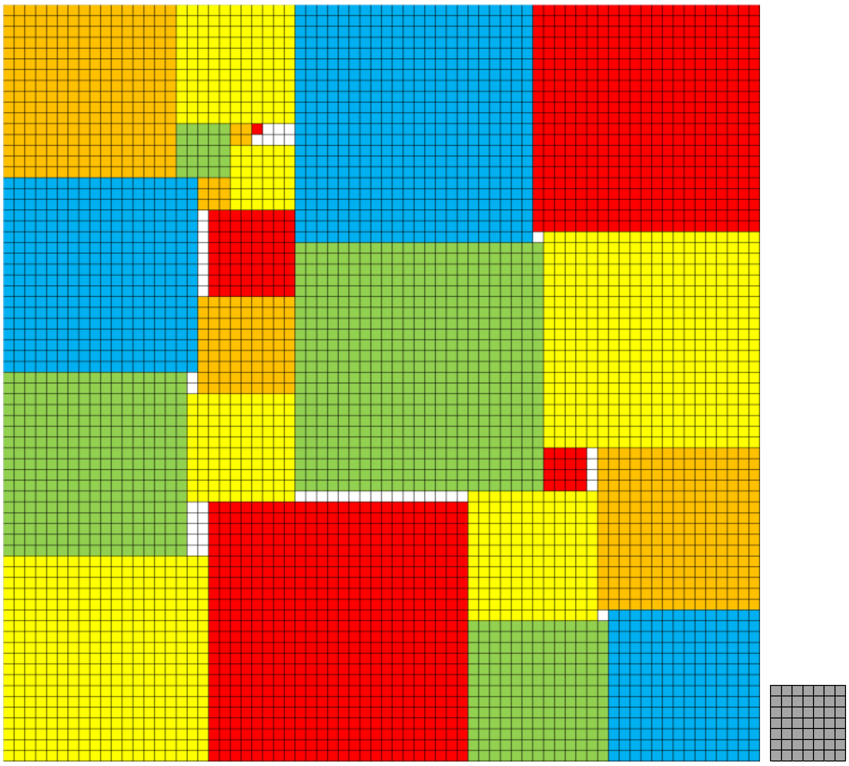

Squares in a Square

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I. Abstract

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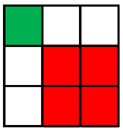
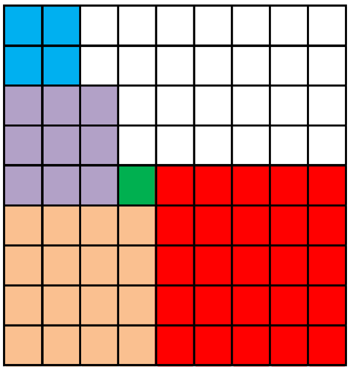
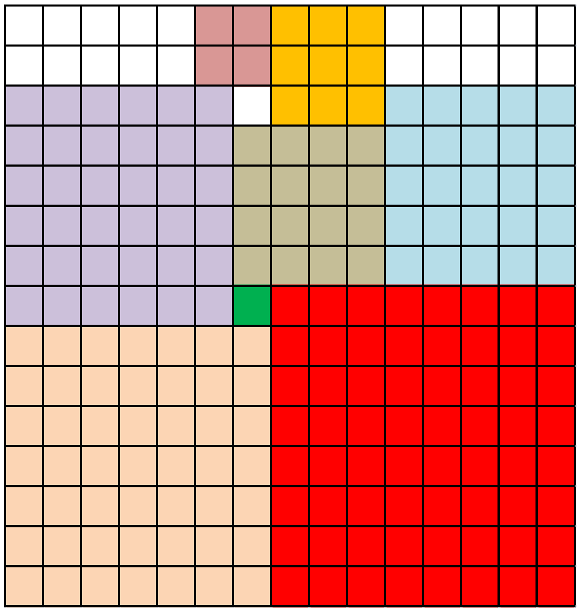


m = 3
n = 1 ... 2
remainder = 4



m = 9
n = 1 ... 5
remainder = 26



m = 15
n = 1 ... 8
remainder = 21 – free area in white!

Squaring a square is the intention to fill up a larger square by a set of smaller squares, such that the remaining open area in the larger square becomes a minimum. Several authors (ref. [1 ... 3]) has given hints how to fill up a given square with edge length m by different rules.

For examples use all squares from edge length 1 to a given number n or use a minimum number of selected smaller squares to fill up the larger one by a nice tiling. Based on this simple geometrical task you will find in your free time a very nice series of paired squares (see ref. [1 ... 3] and attachment 3).

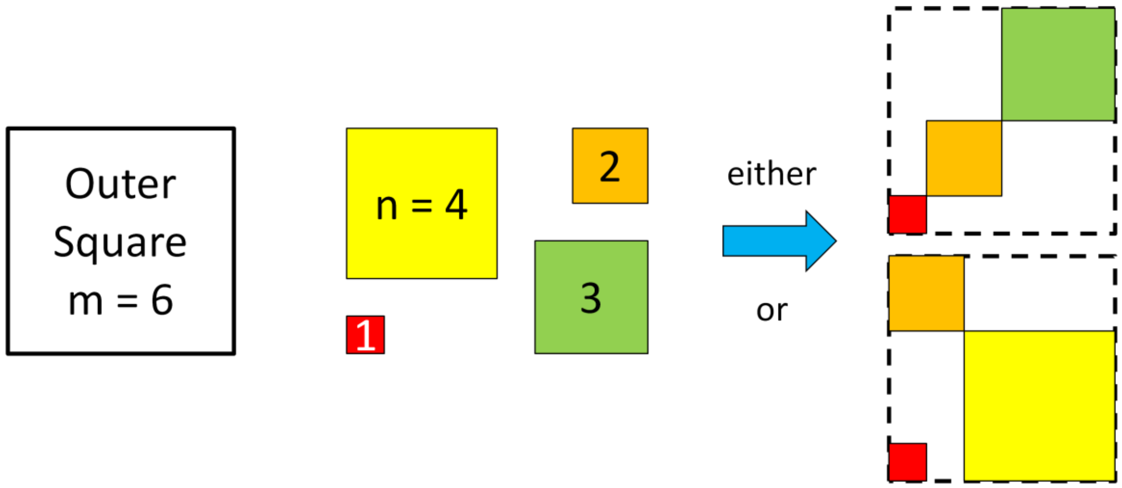
For a given variable k the remaining size/ area (not used by the smaller squares within the larger square with edge length m) will be a minimum. The simple equations which describe this behaviour are in strict contradiction to the huge squares. And more important there are always two neighboured solutions with the same minimum remaining part.

II. Index Terms

Square, tiling, paired solution

III. Squares and more

The tiling of a square with edge length m by several other smaller squares with edge length 1 to n should be the basis for this task. The following examples with four smaller squares with n = 1 ... 4 to be integrated in the larger square with m = 6 is a good basis to explain the full task.



The area of all smaller squares has to be arranged in a larger square with area m^2 in such a way that the remaining part will be a minimum. The above mentioned example with m = 6 and n = 1 ... 4 leads to an outer square of $m^2 = 36$ to be filled by $n^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$. The remaining open part is in theory 6, but nevertheless it is not possible to reach this minimum filling due to the fact that the square with n = 3 or 4 could not be integrated. The remaining part (in white) will be larger and becomes in practice 22 or 15.

The following equation 3.1 describes the squaring of a square with a remaining part.

In fact it is more the task finding the correct sum of square numbers which will be equal with a certain remainder to another "outer" square number.

$$m^2 = \sum_{i=1}^{n} i^2 + remaining\ part = 1^2 + 2^2 + 3^2 + \cdots + n^2 + remaining\ part$$

The first and only perfect square number will be reached for m = 70 and n = 1 ... 24. In this case the remaining part will be zero (because $m^2 = 4900$ and the sum of $1^2 + 2^2 + 3^2 + \ldots + 24^2$ become also 4900). Even if this number seems to be perfect – from calculation point of view – the geometric reality will be different. In practice it is not possible to fill up the larger square m = 70 with all smaller squares from n = 1 ... 24. The best solution could be achieved if you leave the n = 7 square beside (see the example on front page).

For all squares until m = 2 337 238 and n = 1 ... 25 400 there will be no more a perfect solution of a larger square number equal to a sum of smaller square numbers.

IV. Pair of Square numbers

The calculation of the above mentioned equation is very simple and straight forward. Just prepare a table like beside one, use n as variable and calculate the necessary m as indicated in the following steps:

a.) calculate sum of n^2
b.) square root out of sum
c.) round up this m value
d.) size of outer square m^2
e.) and as final result calculate the remaining part of the above mentioned equation.

You will easily find the perfect solution for n = 24

And you will realize that from time to time surprisingly a pair of square number exist where the remaining part is a minimum (as indicated in green).

This minima could be numbered with an index from 1 (starting for the pair with n = 47/48) until k

The remaining part becomes in relationship to the larger square number smaller and smaller.

For example the remaining part of 400 for k = 20st square with an edge length m = 1 536 060 has a relative value of 1,6 x 10^{-10} , which is really a very small value.

Attachment 1 is showing a graph of the remaining open part of square numbers in a larger square number as function of used smaller square number n.

n	Sum of all n^2	Square root Sum	rounded m	Outer Square m^2	Remainder
15	1240	35.214	36	1296	56
16	1496	38.678	39	1521	25
17	1785	42.249	43	1849	64
18	2109	45.924	46	2116	7
19	2470	49.699	50	2500	30
20	2870	53.572	54	2916	46
21	3311	57.541	58	3364	53
22	3795	61.604	62	3844	49
23	4324	65.757	66	4356	32
24	4900	70.000	70	4900	0
25	5525	74.330	75	5625	100
45	31395	177.186	178	31684	289
46	33511	183.060	184	33856	345
47	1. pair	188.997	189	35721	1
48		194.997	195	38025	1
49	40425	201.060	202	40804	379
50	42925	207.183	208	43264	339
189	2268315	1506.093	1507	2271049	2734
190	2304415	1518.030	1519	2307361	2946
191	2. 2340896	1529.999	1530	2340900	4
192	2377760	1541.999	1542	2377764	4
193	2415009	1554.030	1555	2418025	3016
194	2452645	1566.092	1567	2455489	2844
429	26409955	5139.062	5140	26419600	9645
430	26594855	5157.020	5158	26604964	10109
431	3. 26780616	5174.999	5175	26780625	9
432	26967240	5192.999	5193	26967249	9
433	27154729	5211.020	5212	27164944	10215
434	27343085	5229.062	5230	27352900	9815
765	149525115	12228.046	12229	149548441	23326
766	150111871	12252.015	12253	150136009	24138
767	4. 50700160	12275.999	12276	150700176	16
768	51289984	12299.999	12300	151290000	16
769	151881345	12324.015	12325	151905625	24280
770	152474245	12348.046	12349	152497801	23556
1197	572407395	23925.037	23926	572453476	46081
1198	573842599	23955.012	23956	573889936	47337
1199	5. 75280200	23984.999	23985	575280225	25
1200	76720200	24014.999	24015	576720225	25
1201	578162601	24045.012	24046	578210116	47515
1202	579607405	24075.037	24076	579653776	46371
5805	65222528355	255387.017	255388	65223030544	502189
5806	65256237991	255453.005	255454	65256746116	508125
5807	11. 289959240	255519.000	255519	65289959361	121
5808	323692104	255585.000	255585	65323692225	121
5809	65357436585	255651.005	255652	65357945104	508519
5810	65391192685	255717.017	255718	65391695524	502839

Table 1: remaining part as a function of edge length n and m.

V. Solution of "inner" Square numbers in an "outer" Square number

The following table is the result of above demonstrated simple calculation and sum up all the results with k as an index for the paired squares numbers with value n for the smaller inner square numbers and value m for the larger outer square number.

	OEIS						
	A065532	A231174**					
k	n* = n-1	n	m*	m	Remainder		delta m
1	47	**48**	189	195	1		6
2	191	192	1530	1542	4		12
3	431	432	5175	5193	9		18
4	767	768	12276	12300	16		24
5	1199	1200	23985	24015	25		30
6	1727	1728	41454	41490	36		36
7	2351	2352	65835	65877	49		42
8	3071	3072	98280	98328	64		48
9	3887	3888	139941	139995	81		54
10	4799	4800	191970	192030	100		60
11	5807	5808	255519	255585	121		66
12	6911	6912	331740	331812	144		72
13	8111	8112	421785	421863	169		78
14	9407	9408	526806	526890	196		84
15	10799	10800	647955	648045	225		90
16	12287	12288	786384	786480	256		96
17	13871	13872	943245	943347	289		102
18	15551	15552	1119690	1119798	324		108
19	17327	17328	1316871	1316985	361		114
20	19199	19200	1535940	1536060	400		120
21	21167	21168	1778049	1778175	441		126
22	23231	23232	2044350	2044482	484		132
23	25391	25392	2335995	2336133	529		138
24	27647	27648	2654136	2654280	576		144
25	29999	30000	2999925	3000075	625		150

Table 2: Overview of paired square numbers within a larger square number in accordance to equation (3.1)

Surprisingly you will find extremely simple relationship between the index k and all other figures.

I. The missing remaining part is always

$$\text{Remainder} = k^2 \qquad (5.1)$$

II. The outer square number m* and m has obviously a difference of 6k

$$\text{delta } m = 6\,k \qquad (5.2)$$

III. The first outer square number has a value of m, the second one

$$m^* = m - \text{delta } m \qquad (5.3)$$

IV. Per definition (or observation) the smaller inner square number has a value of

$$n^* = n - 1 \qquad (5.4)$$

V. The remaining part is always a constant factor of n

$n = 48$ Remainder

and based on equation 5.1 n will be

$$n = 48\, k^2 \qquad (5.5)$$

VI. The value of the outer square number m will be as follow

with $m^2 = \sum n^2 + \text{Remainder} = \sum n^2 + k^2$

and $m^{*2} = \sum n^{*2}\ \text{Remainder} = \sum n^{*2} + k^2$ with $n^* = n - 1$ and $m^* = m - 6k$

$(m - 6k)^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + k^2$

$m^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 + k^2$

the difference will be

$(m - 6k)^2 - m^2 = - n^2$ with $n = 48\, k^2$ follows

$m^2 - 12\, k\, m + 36\, k^2 - m^2 = - (48\, k^2)^2$

$m = [\, (48\, k^2)^2 + 36\, k^2\,] / 12\, k$

$m = (4^2\, 12)\, k^3 + 3\, k$

and therefore m will be easily calculated as follow

$$m = 3\, k\, (\, 64\, k^2 + 1\,) \qquad (5.6)$$

With these simple equations you are in the position to find easily larger square numbers which will be the sum of smaller square numbers with a minimum remainder.

VI. Summary

The tiling of a larger square with an edge length m by a series of smaller squares from edge length 1 to n was the basis for an interesting search and the behaviour of square numbers.

Based on this simple geometrical task you will find an easy relationship for the k^{th} pair of square number which will lead to a minimum remainder of k^2

The calculation of square numbers leads to four series for n and n* as well as for m* and m
as a function of an index k.

Two of them are already known in OEIS (see ref. 4 and 5), but are calculated on a different basis.

A1: Graph of Remaining part

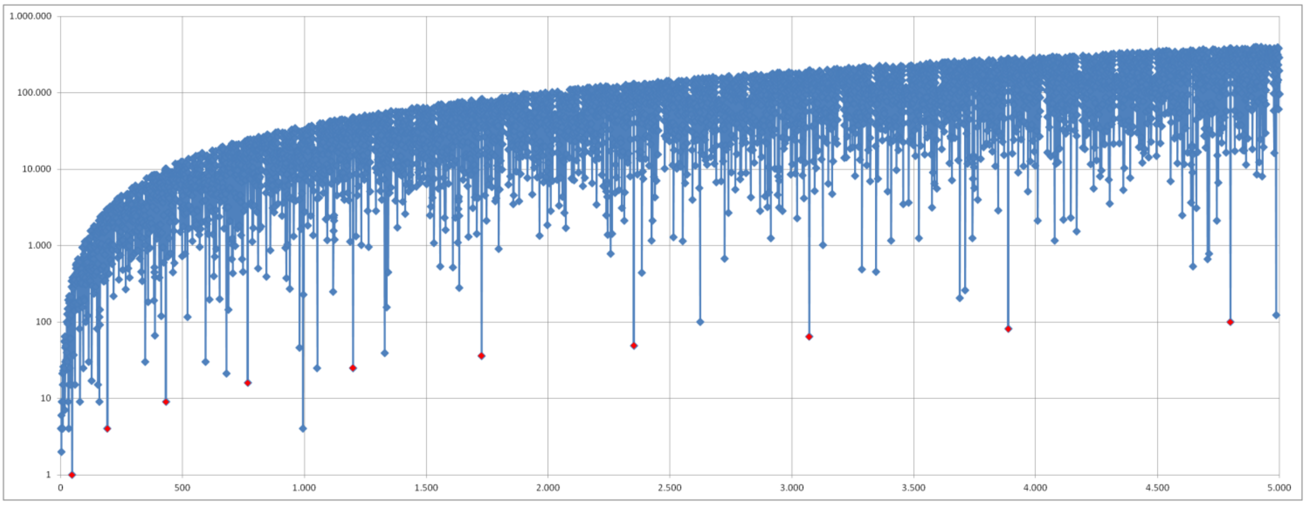


Figure 1: Remainder for certain square number n and m as calculated by equation 3.1

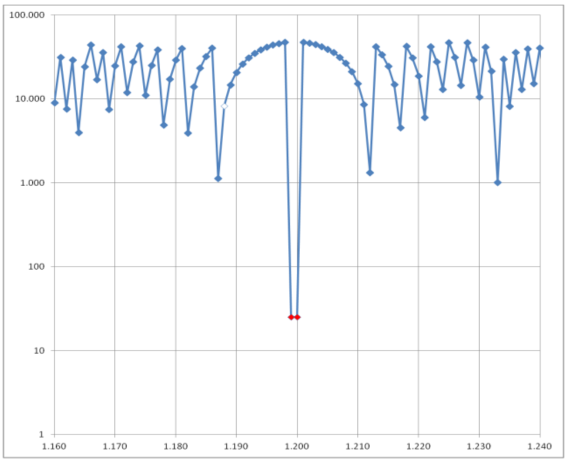


Figure 2: Paired square numbers for n = 1199/ 1200 and the corresponding remaining part of 25.

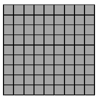
The red marked minima are the solution of equation 3.1 and represents always a pair n; n* and m; m*

A2: References

[1]	https://de.wikipedia.org/wiki/Quadratur_des_Quadrates https://en.wikipedia.org/wiki/Squaring_the_square	General introduction
[2]	Stuart Anderson: *Squared Squares*, 2014. http://www.squaring.net/sq/ss/ss.html	Detailed overview with historical information about different squares
[3]	Martin Gardner, Mathematical Carneval, 1975, Alfred A. Knopf Inc. – New York	Some hints from Martin Gardner
[4]	OEIS at www. Oeis.org A065532 is similar/ equal for n* = n-1	
[5]	OEIS at www. Oeis.org A231174 is similar for n	

A3: One additional example of Squares in a Square

Large outer Square with m = 117
Smaller inner squares with n = 1 ... 34
Remaining part, optimum 4 area units
Remaining part, best 85 area units, without square n = 9, just 0,62% unused area size



square with n = 9 not used