Integers sequence A256881

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Abstract

A256881 is the sequence of the reciprocals of coefficients of a power series arising as the sum of repeated integrals of the hyperbolic sine.

1 A classical result

Let $I = [a, b]$ be an interval such that $0 \in I$ and let $f\colon I \to \mathbb{R}$ be a continuous function.
Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of repeated integrals of f, i.e.

$$\begin{cases} f_0 = f \\ \forall x \in I,\ f_{n+1}(x) = \displaystyle\int_0^x f_n(t)\mathrm{d}t \end{cases}$$

A classical exercice (see e.g. *Le jardin d'Eiden – Une année de colles en Maths Spé*, Eiden Jean-Denis, *Calvage & Mounet*, 2012, p.510) is to prove that:

1. the functions series $\sum f_n$ is normally convergent on I;
2. its sum $F\colon x \mapsto \sum_{n=0}^{+\infty} f_n(x)$ can be written as :

$$\forall x \in I,\ F(x) = f(x) + \int_0^x f(t)\mathrm{e}^{x-t}\mathrm{d}t$$

2 Power Series of the sum of repeated integrals of the hyperboloic sine

If $f = \sinh$, the sum F of its repeated integrals is $F\colon x \mapsto \dfrac{\sinh(x) + x\mathrm{e}^x}{2}$.
F can be written as a power series :

$$F(x) = \sum_{n=1}^{+\infty} \Big(\frac{\delta_n}{2\,n!} + \frac{1}{2(n-1)!}\Big)x^n = \sum_{n=1}^{+\infty} a_n x^n$$

where $\delta_n = 1$ if n is odd and $\delta_n = 0$ if n is even and $a_n = \frac{n+\delta_n}{2\,n!}$.
We state that $\dfrac{1}{a_n} = \dfrac{2 \times n!}{n + \delta_n}$ is equal to $\dfrac{n!}{\lceil n/2 \rceil}$.

Indeed, if $n = 2m$ is even, then $\dfrac{1}{a_n} = \dfrac{1}{a_{2m}} = \frac{(2m)!}{m} = \dfrac{n!}{\lceil n/2 \rceil}$. And if $n = 2m+1$ is odd, then $\dfrac{1}{a_n} = \dfrac{1}{a_{2m+1}} = \frac{(2m+1)!}{m+1} = \dfrac{n!}{\lceil n/2 \rceil}$.

The sequence $\left(\dfrac{1}{a_n}\right)_{n\in\mathbb{N}^*} = (1, 2, 3, 12, 40, 240, 1260, 10080, 72576, \ldots)$ corresponds to the A256881 OEIS sequence.