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THÈSE

PRÉSENTÉE DEVANT

L'UNIVERSITÉ DE BORDEAUX I

POUR OBTENIR LE TITRE DE

DOCTEUR EN INFORMATIQUE

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par

Myriam DE SAINTE-CATHERINE

COUPLAGES ET PFAFFIENS EN COMBINATOIRE, PHYSIQUE ET INFORMATIQUE

Soutenue le** 30 Mars 1983* ***, devant la Commission d'examen :

MM.	*M. MENDES-FRANCE*	....	***Président.***
	H. COHEN	....	*Examinateurs*
	R. CORI	....	
	R. GEORGES	....	
	P. LAFON	....	
	G. VIENNOT	....	

— 1983 —

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Tableau III. 5.

p	0	1	2	3	4	5	6	7	8	9	10	11	12
1		1											
2	1		1										
3		2		1									
4	1		5		1								
5		3		14		1							
6	1		14		42		1						
7		4		84		132		1					
8	1		30		594		429		1				
9		5		330		4719		1430		1			
10	1		55		4719		40898		4862		1		
11		6		1001		81796		379236		16796		1	
12	1		91		26026		1643356		3711916		58786		1
13		7		2548		884884		37119160		37975756		208012	
14	1		140		111384		37119160		922268360		403127256		742900
15		8		5712		6852768		1844536720		24801924512		...	
16	1		204		395352		553361016		105408179176		713055329720		...
17		9		11628		41314284		55804330152		6774025632340		...	
18	1		285		1215126		6018114036		6774025632340		...		...
19		10		21945		20495152		...		...		...	
20	1		385		3331251		51067020290		...		...		...
21		11		38962		869562265		...		...		...	
22	1		506		8321170		...		...		...		...
23		12		65780		...		...		...		...	
24	1		650		19240750		...		...		...		...
[illegible]		13		...		...		...		...		...	
26	1		819		41683005		...		...		...		...
27		14		...		...		...		...		...	
28	1		975		...		...		...		...		...

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Tableau III. 5.

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p \ n	0	1	2	3	4	5	6	7	8	9	10	11	12
1		1											
2	1		1										
3		2		1									
4	1		5		1								
5		3		14		1							
6	1		14		42		1						
7		4		84		132		1					
8	1		30		594		429		1				
9		5		330		4719		1430		1			
10	1		55		4719		40898		4862		1		
11		6		1001		81796		379236		16796		1	
12	1		91		26026		1643356		3711916		58786		1
13		7		2548		884884		37119160		37975756		208012	
14	1		140		111384		37119160		922268360		403127256		742900
15		8		5712		6852768		1844536720		24801924512		...	
16	1		204		395352		553361016		105408179176		713055329720		...
17		9		11628		41314284		55804330152		6774025632340		...	
18	1		285		1215126		6018114036		6774025632340		...		...
19		10		21945		20495152		...		...		...	
20	1		385		3331251		51067020290		...		...		...
21		11		38962		869562265		...		...		...	
22	1		506		8321170		...		...		...		...
23		12		65780		...		...		...		...	
24	1		650		19240750		...		...		...		...
25		13		...		...		...		...		...	
26	1		819		41683005		...		...		...		...
27		14		...		...		...		...		...	
28	1		975		...		...		...		...		...

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$$
\left.
\begin{array}{lll}
\text{Donc} & v = v' R x \bar{x} , & \\
\text{ou} & v = v' R R \bar{x} , & \\
\text{ou} & v = v' x R \bar{x} , & \text{avec } |v'| = n-2 .
\end{array}
\right\} \quad (\text{III. } 20)
$$

Notons $\mathcal{M}^{cm}_{n+1}$ l'ensemble des mots de Motzkin "modifiés", c'est-à-dire qui satisfont (III. 20) .

$$
\begin{array}{ll}
u \in \mathcal{M}^{c}_{n+1} \Rightarrow & u = u' R , \\
& \text{ou } u = u'B , \\
& \text{ou } u = u'\bar{x} , \quad \text{avec } |u'| = n - 1 .
\end{array}
$$

Soit $\varphi : \mathcal{M}^{cm}_{n+1} \to \mathcal{M}^{c}_{n-1}$.

telle que :

$$
\forall w \in \mathcal{M}^{cm}_{n+1} , \ w = w_1 w_2 \dots w_{n+1} , \ \varphi(w) = u = u_1 u_2 \dots u_{n-1} ,
$$

avec
$$
\begin{cases}
\forall i \in [1, n-2] , \ u_1 = w_i . \\
w_{n-1} = x , \ w_n = R , \ w_{n+1} = \bar{x} \Rightarrow u_{n-1} = R , \\
w_{n-1} = R , \ w_n = x , \ w_{n+1} = \bar{x} \Rightarrow u_{n-1} = B , \\
w_{n-1} = R , \ w_n = R , \ w_{n+1} = \bar{x} \Rightarrow u_{n-1} = \bar{x} .
\end{cases}
$$

Il est trivial de vérifier que φ est une bijection (Fin de preuve du lemme III. 19) .

Comme $|\mathcal{M}^{c}_{n-1}| = C_n$ (voir Lemme III. 11) , on en déduit la Proposition III. 17 ∎

Exemple $n = 5$.

$(a_i)_{1 \leq i \leq 3} = (1, 2, 4)$,

$(b_i)_{1 \leq i \leq 2} = (2, 3)$.

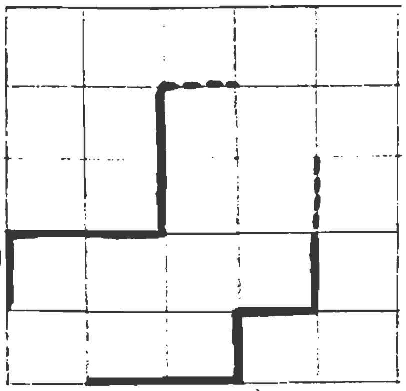

Figure III. 12.

Ajoutons un pas à chaque chemin (voir Figure III. 12) :
un pas Est pour ch(b) et un pas Nord pour ch(a) , de manière à pouvoir associer au couple formé, un mot de Motzkin coloré, par une bijection ψ'' :

Soient $ch(a) = \omega_1 \omega_2 \cdots \omega_n$,
$ch(b) = \omega'_1 \omega'_2 \cdots \omega'_n$, et $\psi''(ch(a), ch(b)) = u = u_1 \ldots u_n$.

$$\left|\begin{array}{l}
\omega_i = E \text{ et } \omega'_i = N \Rightarrow u_i = x , \\
\omega_i = E \text{ et } \omega'_i = E \Rightarrow u_i = B , \\
\omega_i = N \text{ et } \omega'_i = N \Rightarrow u_i = R , \\
\omega_i = N \text{ et } \omega'_i = E \Rightarrow u_i = \bar{x} .
\end{array}\right.$$

L'ajout d'un pas aux chemins revient à concaténer un $\bar{x}$ à u . Soit donc $v = u\bar{x} = v_1 v_2 v_3 \ldots v_{n+1}$.

Remarquons de plus, que les deux derniers pas de ch(b) sont toujours verticaux. Ceci implique :

$$v_{n-1} = x \text{ ou } R ,$$
$$v_n = x \text{ ou } R .$$

(III. 19)
$$\begin{cases} \ell \le n , \\ 1 < a_1 < a_2 < \dots < a_{\ell-1} < a_\ell , \\ 1 < b_1 < b_2 < \dots < b_{\ell-1} , \\ a_i \le b_i \quad \text{pour} \quad i \in [1, \ell-1] , \\ 1 < a_i \le n-2 \quad \text{pour} \quad i \in [1, \ell-1] , \\ a_\ell \le n , \\ 1 < b_i \le n-1 \quad \text{pour} \quad i \in [1, \ell-1] . \end{cases}$$

LEMME III. 19. -

L'ensemble des couples de suites vérifiant (III. 19) est en bijection avec l'ensemble des mots de Motzkin colorés à $n-1$ lettres .

Preuve :

On utilise la bijection ψ' définie entre les couples de suites vérifiant (III. 10) et (III. 11) et certains couples de chemins du plan ne se coupant pas. Rappelons cette bijection ψ' :

Soient $(a_i)_{1 \le i \le \ell}$ et $(b_i)_{1 \le i \le \ell}$ le couple de suites. On associe à chacune des suites , un chemin dans le plan . Soit $ch(a) = \omega_1 \omega_2 \dots \omega_n$ le chemin associé à la suite (a_i) , il est tel que :

$$\forall i \in [1, n] , \exists j \in [1, \ell], i = a_j \Rightarrow \omega_i = E \text{ (pas Est) ,}$$
$$\forall i \in [1, n] , \forall j \in [1, \ell], i \neq a_j \Rightarrow \omega_i = N \text{ (pas Nord) .}$$

On définit de même $ch(b)$.

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[6] C.C. Lindner, A. Rosa, *Steiner quadruple systems - a survey*, Disc. Math. 22 (1978), 147-181.

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COMBINATORIAL VIEW OF THE COMPOSITION OF FUNCTIONS

S. Getu and L.W. Shapiro

Abstract

In this paper a way of picturing the composition, $F(G(x))$ of exponential generating functions is discussed. The special case where $F(x)$ is the exponential function has been discussed before many times. See for instance the survey article by Stanley [12], the monograph by Moon [6] and Riordan's books [9],[10] and the references there. The simplicity involved however seems to get lost in such complications as Bell polynomials and Faa di Bruno's formula. The purpose there is to discuss the method and to give a small selection of results that can then be obtained. Some of the results are well known but some such as the combinatorial interpetations of the Hermite and Laguerre polynomials are of independent interest.

We have only begun to list the results that can be viewed this way but hope that many readers will find this pictorial method personally useful.

§1. *Some standard generating functions and what they look like.*

If $a_0, a_1, a_2, a_3, \ldots$ is a sequence then the formal series

$$A(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$ is the

exponential generating function (or E.G.F.) for this sequence. In this paper a_n will be the number of ways that a set with n elements can be arranged according to some conditions. This is illustrated by the following eight examples. Proofs can be found in [12], or to some extent in [2] or [6].

(A) Just count each set once. This gives the E.G.F.

$$1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^4}{4!} + \ldots = e^x$$

In this case we are putting no structure on the set.

(B) There are $n!$ permutations of a set with n elements. This yields the E.G.F.

$$1 + 1!\, x + 2!\, \frac{x^2}{2!} + 3!\, \frac{x^3}{3!} + 4!\, \frac{x^4}{4!} + \ldots = \frac{1}{1-x} = P(x).$$

There is an alternate view that is helpful. Each permutation can be written as a product of disjoint cycles in essentially unique way. These cycles partition the n-set. Call each subset involved in the partition a block. The conditions could be specified for the block instead of for the whole set. In this example we could specify that each block consists of elements of some cyclic permutation.

(C) Rooted trees.

$$1x + 2\cdot\frac{x^2}{2!} + 3^2\cdot\frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} n^{n-1}\frac{x^n}{n!} = T(x)$$

It is well known that $T(x) = xe^{T(x)}$.

(D) n-cycles. (There is only one block and it is an n-cycle.)

$$x + 1\cdot\frac{x^2}{2!} + 2!\,\frac{x^3}{3!} + 3!\,\frac{x^4}{4!} + \dots = -\ln(1-x) = N(x)$$

(E) Idempotent functions with a single root

$$1\cdot x + 2\cdot\frac{x^2}{2!} + 3\,\frac{x^3}{3!} + \dots = xe^{x} = I(x)$$

(F) All functions from $[n]$ to $[n]$. (i.e. all functional digraphs on $[n]$)

$$1 + 1\cdot x + 2^2\,\frac{x^2}{2!} + 3^3\,\frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} n^n\frac{x^n}{n!} = A(x)$$

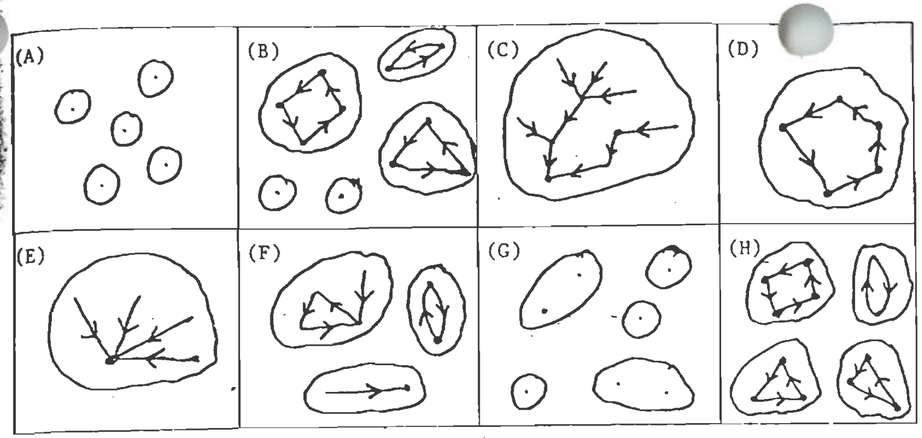
(G) Let S be an n-set. In how many ways can S be partitioned into subsets where each subset consists of one or two elements? Let the number of such possibilities be s_n and let $S(x) = \exp(x+\frac{x^2}{2}) = \sum_{n\geq 0}^{\infty} s_n\frac{x^n}{n!}$ be the exponential generating function. This is the number of ways that n subscribers of a telephone exchange can be connected since if two people are talking on the telephone they make up a subset where if someone is not talking on the telephone that person comprises a singleton subset.

This function also enumerates the number of elements of order 1 or 2 in the symmetric group S_n and the number of symmetric permutation matrices.

(H) The derangements of n elements. Every block is an n-cycle where $n \geq 2$.

$$D(x) = \frac{1}{1-x}e^{-x}$$

Typical pictures are given



§2. *Composition of Functions.*

The next idea we want to discuss is the composition of generating functions. The basic idea is very simple if illustrated by pictures. See [2], [8], [9] and [10] for a more formal and detailed discussion.

Let $F(x)$ and $G(x)$ be exponential generating functions. If $F(x) = \sum_{n\geq 0} f_n\frac{x^n}{n!}$ and $G(x) = \sum_{n\geq 1} g_n\frac{x^n}{n!}$ then what is the meaning of $F(G(x))$ as an exponential function? More specifically, what does the kth term $f_k\frac{(G(x))^k}{k!}$ represent? Think of k vertices arranged in one of the configurations enumerated by f_k. Then enlarge each vertex to circle the contents of which are enumerated by $G(x)$.

Let us consider $G(x)^2$ in some detail.

$$G(x)^2 = \left(\sum_{n=0}^{\infty} g_n\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} g_m\frac{x^m}{m!}\right) = \sum_{\ell=0}^{\infty}\cdot\sum_{n=0}^{\ell}\binom{\ell}{n} g_n\, g_{\ell-n}\frac{x^\ell}{\ell!}\ .$$

So to account for ℓ vertices we put n vertices in one group, $\ell - n$ in the other. There are then g_n ways to arrange the first group, $g_{\ell-n}$ ways for the second. However our calling one group the first and the other the second is arbitrary so $(G(x))^2/2!$ is the term we want.

Similarly $\frac{(G(x))^k}{k!}$ will be the term when k vertices are replaced by configurations each enumerated by $G(x)$.

To illustrate this consider the following:

EXAMPLE

What is the interpretation of

$$P(T(x)) = \frac{1}{1 - T(x)}$$

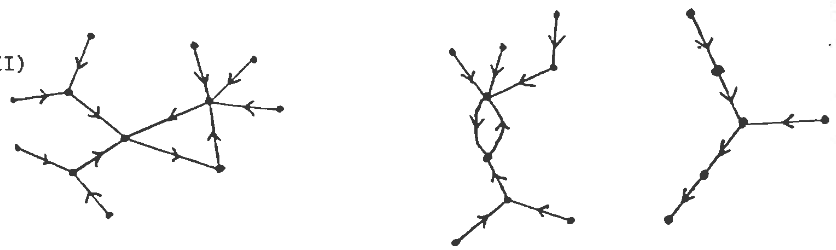
We start by considering a typical permutation

(I) 

We then think of

(II) 

Since $T(x)$ represents a rooted tree we can go another step and thinking of each $T(x)$ as some typical rooted tree we can then identify the root to the vertex in (I) to get the following picutre

(III) 

We then recognize this as the picture of the typical funtional digraph so

$$A(x) = \frac{1}{1 - T(x)}$$

$$\begin{aligned} A(x) &= 1 + x + 2^2 \frac{x^2}{2!} + 3^3 \frac{x^3}{3!} + 4^4 \frac{x^4}{4!} + \ldots \\ &= 1 + (x + 2\frac{x^2}{2!} + 3^2 \frac{x^3}{3!} + 4^3 \frac{x^4}{4!} + \ldots) \\ &\quad + (x + 2\frac{x^2}{2!} + 3^2 \frac{x^3}{3!} + \ldots)^2 \\ &\quad + (x + 2\frac{x^2}{2!} + \ldots)^3 \\ &\quad + (x + \ldots)^4 \end{aligned}$$

From this we can see that

$$n^n = \sum_{n_i \geq 1} \binom{n}{n_1, \ldots, n_k} n_1^{\,n_1 - 1}\, n_2^{\,n_2 - 1} \cdots n_k^{\,n_k - 1}$$

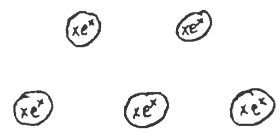
where the sum is taken over all compositions of n.

EXAMPLE 2.

How many idempotent functions on $[n]$ are there? We start with

(I)  $\longleftrightarrow e^x$

(II) Replace each vertex by xe^x



Again each xe^x represents a rooted connected idempotent so we can identify the root with our original vertex in (I) to obtain the picture we want

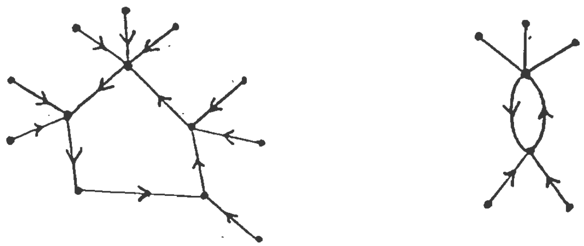
This has e^{xe^x} as its exponential generating function which is what we wanted.

These last two examples yielded well known results but the methods apply much more generally.

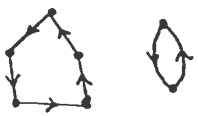
EXAMPLE 3.

What is the exponential generating function for funtional digraphs such that each element either is part of a cycle or at length at most 1 from a cycle?

The picture is



which can be built up by starting with (I)

 $\longleftrightarrow \dfrac{1}{1 - x}$

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and adding  $\longleftrightarrow$ xe^x so the answer is $\dfrac{1}{1 - xe^x}$.

The next theorem is an important one in examining the symmetric semigroup of all functions from [n] to itself. Call this semigroup $\bar{S}_n$. It is easy to see that any element of this semigroup acts as a permutation on some k elements and maps the remaining $n - k$ elements eventually to the first k. We call the first k elements the permutation part.

We have some condition that we want on the permutation part. Possibilities include, a) no cycles of length 1 b) all cycles have odd length c) the permutation part is even (in the alternating group). If there are b_k arrangements on k elements then let

$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^k}{k!}$ (usually we would have $B(x)$ in closed form).

We now want to consider the whole of $\bar{S}_n$ where the permutation part satisfies the given condition. Geometrically we now attach trees to the permutation part. Let b_n^* be the number of elements of $\bar{S}_n$ whose permutation part satisfies the given condition. Let

$B^*(x) = \sum_{n=1}^{\infty} b_n^* \frac{x^n}{n!}$. What is the relationship of $B^*(x)$ and $B(x)$?

The answer is simple.

THEOREM: *With this notation*

$$B^*(x) = B(T(x))$$

Proof. The proof merely generalizes example 1 since $\frac{1}{1 - x}$ is the E.G.F. for all permutations (i.e. with no additional condition on the permutation part). Since we can attach a tree to any of the elements in the permutation part the appropriate E.G.F. is $B(T(x))$.

The number of labelled rooted forests with k specified roots is kn^{n-k-1} and proofs can be found in [2], [3] and [7]. Thus

$b_n^* = \sum_{k=1}^{n} \binom{n}{k} b_k kn^{n-k-1}$. We choose the k points [in $\binom{n}{k}$ ways], arrange them [in b_k ways] and attach the rooted trees [in kn^{n-k-1} ways].

EXAMPLE 4.

If we let n_k be the number of k-cycles on k vertices as in (D), then

$$N(x) = -\ln(1 - x) \text{ and}$$
$$N^*(x) = -\ln(1 - T(x))$$

Expanding would yield $n_m^* = \sum_{k=0}^{m} \binom{m}{k} (k - 1)!\, kn^{n-k-1}$ and if $m \geq 3$ the number of unicyclic simple graphs on m points is $\frac{n_m^*}{2}$.

EXAMPLE 5.

One presentation of the Fibonacci numbers is as the number of compositions of n into odd parts. For instance 6 has the following eight compositions.

$5 + 1,\ 1 + 5,\ 3 + 3,\ 3 + 1 + 1 + 1,\ 1 + 3 + 1 + 1,\ 1 + 1 + 3 + 1,$

$1 + 1 + 1 + 3,\ 1 + 1 + 1 + 1 + 1 + 1.$

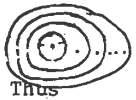
For sets we could ask for the number of chains of subsets

$S_0 = \phi \subset S_1 \subset S_2 \subset S_3 \subset \ldots \subset S_k = S$ where $|S_i - S_{i-1}|$ is odd $i = 1,2,\ldots,k$ and $|S| = n$. Call this number $\tilde{F}_n$. It is easy to see that $\tilde{F}_1 = 1$, $\tilde{F}_2 = 2$, $\tilde{F}_3 = 7$, and $\tilde{F}_4 = 32$. What is

$\sum_{n=0}^{\infty} \tilde{F}_n \frac{x^n}{n!}$. First arrange a set with k elements in a row and allowing the empty set this has E.G.F.

$$1 + x + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \ldots = \frac{1}{1 - x}$$

This picture is  or by taking unions from the left.

 Then replace each vertex by $G(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sinh x$. Thus

$$\sum_{n=0}^{\infty} \tilde{F}_n \frac{x^n}{n!} = \frac{1}{1 - \sinh x} = 1 + x + 2\frac{x^2}{2!} + 7\frac{x^3}{3!} + 32\frac{x^4}{4!} + 181\frac{x^5}{5!} +$$

$$1232\frac{x^6}{6!} + 9787\frac{x^6}{7!} + \ldots$$

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If instead we had required $|S_i - S_i - 1| \geq 2$ we would have $G(x) = e^x - 1 - x$ and E.G.F.

1, 2, 9, 61, 551, 6221, 84285, ...

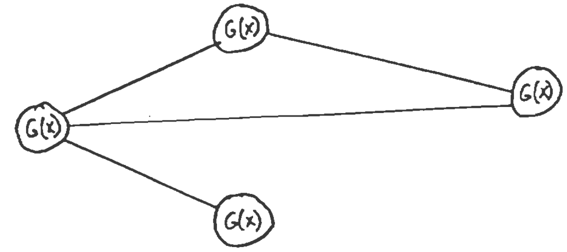
$$\frac{1}{1 - (e^x - 1 - x)} = \frac{1}{2 - x - e^x}$$

for another similar analogue of the Fibonacci numbers. See Gross [5] where this situation arises in relation to preferential arrangements.

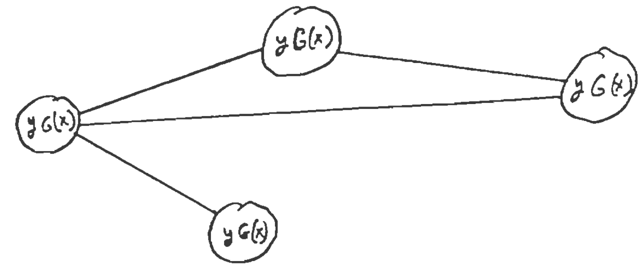
Several remarks are in order here. First when considering $A(B(x))$ the power series for $B(x)$ should have no constant term. Second if $C(x)$ enumerates some unrooted configurations then $x\frac{d}{dx}(C(x))$ enumerates the rooted version of the same configurations. The rooted version has a root which may be attached to the vertex in the diagram of part I.

§3. *Combinatorial polynomials.*

There is a simple way of pushing these illustrations one step further. When we have state II



imagine each circle to be colored one of y colors. This yields the following picture:



The kth term becomes $f_k \frac{(G(x))^k y^k}{k!}$ and the new composite function is $F(yG(x))$.

indicating the number of blocks or components involved. Two familiar examples of this are:

EXAMPLE 6.

Let each block consist of one element that can be colored y ways. This yields

$$e^{yx} = \sum_{n=0}^{\infty} y^n \frac{x^n}{n!}$$

EXAMPLE 7.

Let the set N be broken up into blocks and let each block be colored one of y colors. A single block is then enumerated by

$$y(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) = y(e^x - 1)$$

and the exponential generating function is

$$e^{y(e^x - 1)}$$

However $[n]$ can be partitioned into k subsets in $S(n,k)$ ways where $S(n,k)$ is a Stirling number of the second kind. Thus

$$e^{y(e^x - 1)} = 1 + \sum_{n=0}^{\infty} \sum_{k=0} S(n,k) y^k \frac{x^n}{n!}$$

Letting $y = 1$ yields the familiar

$$e^{e^x - 1} = 1 + \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$$

where $B(n) = \sum_{k=1}^{n} S(n,k)$ is the nth Bell number, the total number of blocks on N.

The next example provides a connection with Hermite polynomials.

EXAMPLE 8.

Let $[n]$ be partitioned so that each block consists of 1 or 2 elements. Those with one element can be colored $2y$ ways while those with two elements can be oriented towards either vertex. A picture might look like

On one hand the exponential generating function is

$$e^{2yx + 2\frac{x^2}{2}}$$

On the other hand the coefficient polynomial for $x^n/n!$ is

$$\sum_{k\geq 0} \underbrace{\binom{n}{2k}}_{\substack{\text{pick } 2k \text{ element}\\ \text{for the pairs}}} \cdot \underbrace{(2y)^{n-2k}}_{\substack{\text{each singleton}\\ \text{can be colored}\\ 2y \text{ ways}}} \cdot \underbrace{\binom{2k}{2,2,2,\ldots,2}\frac{1}{k!}}_{\substack{\text{pair the}\\ \text{elements}}} \cdot \underbrace{2^k}_{\substack{\text{orient}\\ \text{each pair}}}$$

which simplifies to

$$\sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{(n-2k)!} \frac{(2y)^{n-2k}}{k!}$$

For $n = 5$ this yields

$$32y^5 + 160y^3 + 120y.$$

This is essentially the $H_5(Y)$, the fifth Hermite polynomial except that the signs do not alternate. Indeed this is what is happening and replacing x by it and y by $-iz$ yields

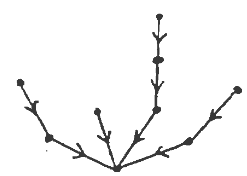
$$\exp(2zt - t^2) = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}$$

which is the standard generating function for the Hermite polynomials. Thus we have a very pleasant, down to earth, combinatorial view of the Hermite polynomials. A recent paper of Foata's [4] takes this idea one step further and proves Mehler's identity for Hermite polynomials

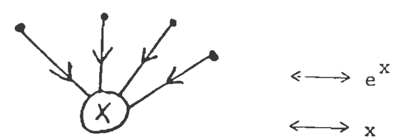
$$1 + \sum_{n\geq 1} H_n(a) H_n(b) \frac{u^n}{n!} = (1 - 4u^2)^{-\frac{1}{2}} \exp\left[\frac{4abu - 4(a^2 + b^2) u^2}{1 - 4u^2}\right]$$

§4. *Linear trees and the Laguerre polynomials.*

Define a linear tree to be a rooted tree where only the root can have degree greater than two. The picture is



Recall that the E.G.F. for a connected idempotent graph is xe^x where x gives the root and e^x the other vertices



Replacing each of the other vertices by a rooted linear graph with E.G.F. $\frac{x}{1-x}$ gives the picture for linear trees and thus the E.G.F. is

$$G(x) = xe^{\frac{x}{1-x}} = x\exp\left(\frac{x}{1-x}\right).$$

The first few values are

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g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	2	9	52	365	3006	28357	301064

and the recursion is $(n-1)g_n = (2n^2 - 3n)g_{n-1} + (n^2 - n)(3-n)g_{n-2}$. If the root is omitted then we obtain

$$G^*(x) = e^{\frac{x}{1-x}} = 1 + x + 3\frac{x^2}{2!} + 13\frac{x^3}{3!} + 73\frac{x^4}{4!} + 501\frac{x^5}{5!} + 4051\frac{x^6}{6!} + \ldots$$

which is Sloane's sequence 1190 where it is listed as forests of greatest height. It is also true that

$$g^*_n = \sum_{k=1}^{n-1} \cdot \frac{n!}{k!}\binom{n-1}{k-1}$$

The numbers $\frac{n!}{k!}\binom{n-1}{k-1}$ are called the (signless) Lah numbers. If each branch out of the root is colored one of z colors, then we have

$$e^{\frac{zx}{1-x}} = \sum_{k=1}^{n-1} \frac{n!}{k!}\binom{n-1}{k-1} z^k \frac{x^n}{n!}$$

and the polynomials in z turn out to be Laguerre polynomials with

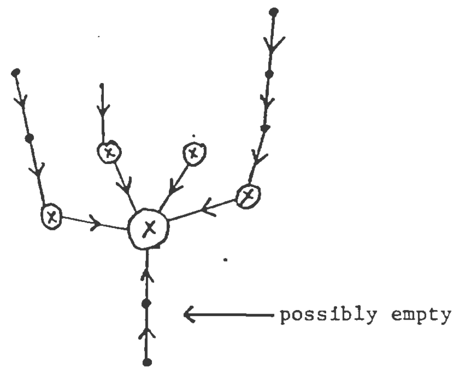
$L^*_n(-z) = \sum_{k=1}^{n-1} \frac{n!}{k!}\binom{n-1}{k-1} z^k$. This striking combinatorial interpretation has been discussed before in Riordan [10] and by Mullin and Rota [8]. As striking as this is, it is not fully satisfactory since it is the $\alpha = -1$ version of the Laguerre polynomials and not the standard version. In a later paper Rota, Kahaner, and Odlyzko [11] do consider complete families of Hermite and Laguerre polynomials from the view point of operators.

The generating function for the standard Laguerre polynomials is

$\frac{1}{1-x} e^{\frac{-zx}{1-x}} = \sum_{n=0}^{\infty} - L_n(z)x^n$. Thus

$$\frac{1}{1-x} e^{\frac{zx}{1-x}} = \sum_{n=0}^{\infty} [n!\, L_n(-z)] \frac{x^n}{n!}$$

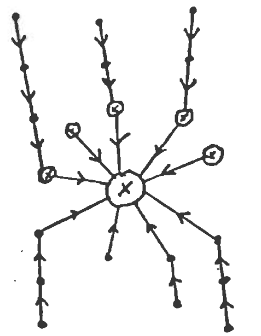
To interpret the polynomials $n!\,L_n(-z)$ start with the rooted idempotent E.G.F. xe^x. Replace the root given by x by a rooted linear tree which is allowed to be empty so that $\frac{1}{1-x}$ is the E.G.F. As before the remaining points given by e^x are each replaced by z colored rooted linear nonempty trees.



Thus $n!\,L_n(-z)$ gives the number of such arrangements on a total of n vertices. This characterization can be used to prove vaious identities involving Laguerre polynomials. In fact the general Laguerre polynomials of index α can be given a similar interpretation. Using the notation of Abramovitz-Stegun [1] one obtains

$$\frac{1}{(1-y)^{\alpha+1}} \exp\left(\frac{-xz}{1-x}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)$$

We illustrate for $\alpha = 3$ where we have $\alpha + 1 = 4$ linear trees coming in from the bottom



Among the identities that can be proven using this combinatorial method are the following as given in Abramowitz-Stegun [1]

1) $H_n(x + y) = \frac{1}{2^{n/2}} \sum_{k=0}^{n} \binom{n}{k} H_k(\sqrt{2}\, x) H_{n-k}(\sqrt{2}\, y)$

2) $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$

3) $L^{(\alpha+\beta+1)}(x + y) = \sum_{k=0}^{n} L_k{}^{(\alpha)}(x) L_{n-k}{}^{(\beta)}(y)$

We will prove the second of these. Let $H^*_{n+1}(y)$ be the number of ways to partition $\{1,2,\ldots,n + 1\}$ into doubletons and singletons as before with the doubletons with an arrow and the singletons colored any of $2y$ colors. Look at the point $n + 1$. If it is a singleton it is one of $2y$ colors and the remaining n points allow $H^*_n(y)$ possibilities. Otherwise $n + 1$ is a doubleton. There are n choices for a companion, 2 ways to orient the arrow, and $H^*_{n-1}(y)$ possibilities for the other elements. Thus $H^*_{n+1}(y) = 2y\, H^*_n(y) + 2n\, H^*_{n-1}(y)$.

Substituting iz for y now leads to $H^*_{n+1}(z) = 2zH^*_n(z) - 2nH^*_{n-1}(z)$.

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