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THE NUMBER OF CONJUGACY CLASSES OF THE ALTERNATING GROUP OF DEGREE n

by Robert D. Girse

For $n \geq 2$, let $g(n)$ denote the number of conjugacy classes of the alternating group of degree n, and let $g(0) = g(1) = 2$. The purpose of this paper is to give two new formulas for computing $g(n)$, and using them construct a table of values of $g(n)$ up to $n = 200$.

Denes, Erdos and Turan [1, p. 95] prove

$$g(n) = \tfrac{1}{2}(p(n) + 3Q(n)), \qquad (1)$$

where $p(n)$ denotes the number of unrestricted partitions of n, and $Q(n)$ denotes the number of partitions of n into distinct odd parts. Using this result they give a formula for computing $g(n)$ in terms of $p(n)$ alone, [1, p. 90].

Here, using (1), we establish the following results.

THEOREM 1.

$$g(n) = 2\,p(n) + 3 \sum_{r=1}^{[\sqrt{\frac{n}{2}}]} (-1)^r\, p(n-2r^2),$$

where $[x]$ denotes the greatest integer function.

THEOREM 2. For $n > 0$,

$$\sum_r (-1)^r\, g(n-\tfrac{1}{2}(3r^2 \pm r)) = \begin{cases} (-1)^t 3 & \text{if } n = 2t^2 \text{ for } t = 1,2,\ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is taken over all nonnegative integers r for which the argument of g is nonnegative.

To prove the theorems we need the following lemma, in which we give a generating function for $g(n)$.

LEMMA.

$$\sum_{n=0}^{\infty} g(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}\left(2 + 3 \sum_{r=1}^{\infty} (-1)^r x^{2r^2}\right).$$

PROOF. Let $p_2^e(n)$ ($p_2^o(n)$) denote the number of partitions of n into an even (odd) number of parts divisible by 2, and let $\Delta_2(n) = p_2^e(n) - p_2^o(n)$. Now from [2, p. 70] $\Delta_2(n) = Q(n)$, and so (1) can be written as

$$g(n) = \tfrac{1}{2}\,(p(n) + 3\,\Delta_2(n)). \tag{2}$$

The generating functions of p(n) and $\Delta_2(n)$ are given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \tag{3}$$

and

$$\sum_{n=0}^{\infty} \Delta_2(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \prod_{m=1}^{\infty} \frac{1-x^{2m}}{1+x^{2m}}.$$

Thus from (2) we have

$$\sum_{n=0}^{\infty} g(n)x^n = \tfrac{1}{2}\left(\sum_{n=0}^{\infty} p(n)x^n + 3\sum_{n=0}^{\infty} \Delta_2(n)x^n\right)$$

$$= \tfrac{1}{2}\left(\prod_{n=1}^{\infty} \frac{1}{1-x^n} + 3\prod_{n=1}^{\infty} \frac{1}{1-x^n} \prod_{m=1}^{\infty} \frac{1-x^{2m}}{1+x^{2m}}\right)$$

$$= \tfrac{1}{2}\prod_{n=1}^{\infty} \frac{1}{1-x^n}\left(1 + 3\prod_{m=1}^{\infty} \frac{1-x^{2m}}{1+x^{2m}}\right)$$

$$= \tfrac{1}{2}\prod_{n=1}^{\infty} \frac{1}{1-x^n}\left(1 + 3\sum_{r=-\infty}^{\infty} (-1)^r x^{2r^2}\right),$$

where the last equation follows as a special case of Jacobi's identity [4, p. 282]. Simplifying this last expression proves the lemma.

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Theorem 1 follows from the lemma by substituting the series in (3) for the product in the generating function of g(n), multiplying and equating coefficients. To prove Theorem 2, we multiply both sides of the generating function in the lemma by $\prod_{n=1}^{\infty} (1-x^n)$, apply Euler's Pentagonal Number Theorem [4, p. 284] and equate coefficients.

The table that follows was computed using both of the formulas given above to ensure accuracy. Values of the partition function p(n) were taken from [3] as needed.

n	g(n)	n	g(n)
-	-	16	123
2	1	17	156
3	3	18	200
4	4	19	254
5	5	20	324
6	7	21	408
7	9	22	513
8	14	23	641
9	18	24	804
10	24	25	997
11	31	26	1236
12	43	27	1526
13	55	28	1883
14	72	29	2308
15	94	30	2829

n	g(n)
31	3451
32	4209
33	5109
34	6194
35	7485
36	9038
37	10871
38	13063
39	15654
40	18738
41	22365
42	26665
43	31716
44	37682
45	44669
46	52887
47	62494
48	73767
49	86902
50	102260
51	120132
52	140970
53	165153
54	193277
55	225854
56	263647

n	g(n)
57	307329
58	357877
59	416198
60	483547
61	561087
62	650432
63	753132
64	871229
65	1006720
66	1162228
67	1340347
68	1544409
69	1777750
70	2044596
71	2349258
72	2697098
73	3093596
74	3545545
75	4059984
76	4645459
77	5310902
78	6067111
79	6925423
80	7899414
81	9003413
82	10254449

n	g(n)
83	11670643
84	13273336
85	15085276
86	17133170
87	19445635
88	22055973
89	24999995
90	28319236
91	32058464
92	36269335
93	41007664
94	46337581
95	52328597
96	59060224
97	66618714
98	75102497
99	84618575
100	95288507
101	107244643
102	120636996
103	135629035
104	152405517
105	171167962
106	192143553
107	215580390
108	241757455

n	g(n)
109	270979486
110	303588581
111	339958692
112	380508578
113	425696222
114	476033662
115	532081020
116	594463415
117	663864830
118	741047378
119	826845203
120	922186258
121	1028086111
122	1145673167
123	1276182520
124	1420984380
125	1581578544
126	1759626979
127	1956948610
128	2175556646
129	2417654178
130	2685676867
131	2982289921
132	3310436680
133	3673337073
134	4074543782

n	g(n)	n	g(n)
135	4517942683	161	59079619598
136	5007816766	162	64957041669
137	5548849736	163	71399591520
138	6146199495	164	78459835617
139	6805504979	165	86195002642
140	7532970635	166	94667518514
141	8335377742	167	103945322260
142	9220181421	168	114102483824
143	10195527881	169	125219585391
144	11270365570	170	137384436983
145	12454469233	171	150692535352
146	13758568475	172	165247890325
147	15194380254	173	181163576841
148	16774756214	174	198562690888
149	18513726317	175	217579009409
150	20426668712	176	238358096453
151	22530365838	177	261058090946
152	24843200384	178	285850985994
153	27385227031	179	312923567966
154	30178398299	180	342478895505
155	33246655679	181	374737414821
156	36616189642	182	409938672317
157	40315553357	183	448342636636
158	44375963702	184	490231678247
159	48831442120	185	535912135678
160	53719161297	186	585716605571

n	g(n)	n	g(n)
187	640005791947	194	1183011735763
188	699171155543	195	1290420487092
189	763637094903	196	1407285890850
190	833864009926	197	1534415353304
191	910350871632	198	1672683423513
192	993638763523	199	1823036666190
193	1084313902557	200	1986499984086

REFERENCES

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2. R. D. Girse, Certain Identities in Partition Theory, Ph.D. dissertation, Kansas State University, 1979.

3. H. Gupta, A. E. Gwyther and J. C. P. Miller, Tables of Partitions. University Press, Cambridge, 1958.

4. G. H. Hardy and E. M. Wright, Introduction to the Theory of Numbers, 4th edition. Clarendon Press, Oxford, 1968.

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July 29, 1980

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Dear Dr. Sloane,

Enclosed find a preprint of a manuscript which is currently under review by BIT. I hope you will find the sequence of integers in the table a useful addition to your new book.

Sincerely,

Robert D. Girse

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August 12, 1980

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Dear Dr. Girse:

Thank you very much for your letter of July 29. Actually your sequence g(n) is already in the Handbook of Integer Sequences - it is sequence 910. This points to a reference not mentioned by you:

Canadian Jnl. Math.
Vol. 4 (1952) p. 383.

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Yours sincerely,

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