AFTER having collected in Part I[8] some exact information concerning the partition function f of the two-dimensional Ising model we wish to present in this paper some approximate methods of our own and compare their results with the exact information available and other well-known approximate schemes.

Before proceeding let us recall the notation. We denoted by λ the partition function per spin, i.e., for N spins we have

$$f=\lambda^N(K). \tag{37}$$

The parameter K is the only variable on which λ depends. It combines the coupling energy J and the temperature T in the form

$$K=J/2kT. \tag{38}$$

The knowledge of $\lambda(K)$ is not sufficient for the computation of the magnetic properties of the model, but it permits calculation of the thermal quantities, particularly the total energy E and the molar specific heat C [Eqs. (17) and (18)].

5. Power Series Developments of λ

The energy of our system can be obtained by elementary reasoning in both the very high and very low temperature region.

At high temperatures (i.e., $K=0$) the spins orient themselves at random regardless of coupling forces. We conclude, therefore, by direct inspection of Eq. (1) that

$$E(0)=0.$$

By a similar reasoning we find for large K

$$E(\infty)=-NJ.$$

Equations (2) and (37) then permit the computation of λ in these two extreme cases. We find

$$\lambda(0)=2, \tag{39}$$

$$\lambda(\infty)\approx e^{2K}. \tag{40}$$

Either one of these two limiting formulas can be continued by a power series. The continuation of (39) is the well-known development of λ in powers[9] of $1/T$ which in our case means powers of K. If we carry out this development in Eq. (2) we get

$$f=\sum_{\mu_i=\pm 1}\Big[1+K\sum_{\langle i,k\rangle}\mu_i\mu_k+\tfrac{1}{2}K^2\Big(\sum_{\langle i,k\rangle}\mu_i\mu_k\Big)^2+\cdots$$

$$=2^N\Big[1+K\Big\langle\sum_{\langle i,k\rangle}\mu_i\mu_k\Big\rangle_{\text{Av}}+\tfrac{1}{2}K^2\Big\langle\Big(\sum_{\langle i,k\rangle}\mu_i\mu_k\Big)^2\Big\rangle_{\text{Av}}+\cdots.$$

The averages are quite elementary to evaluate because they are to be taken at infinite temperature, that is, regardless of coupling. They are expressions containing various powers of N. But when we raise f to the power $1/N$ in accordance with (37) these powers disappear. Thus we find for λ

$$\lambda=2\left(1+K^2+\frac{4}{3}K^4+\frac{77}{45}K^6+\frac{1009}{315}K^8+\cdots\right). \tag{41}$$

[8] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941), this issue.

[9] W. Opechowski, Physica **4**, 181 (1937).