Counting Distinct Binary Arrays With Respect To Isometric Transformations

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Abstract

In this paper we give a method for calculating the number of distinct binary arrays of size m by n with respect to isometric transformations (rotations, reflections and translations). Using logic which allows us to ignore translations, we then use Burnside's lemma and partition the arrays in terms of the size of their bounding box in order to count the distinct arrays.

1 Introduction

An m by n binary array has 2^{mn} possible states. Not all states are unique however. For example,

00
11

May via an isometry be transformed to

01
01

and hence the two are considered the same.

For the rest of this paper, without loss of generality we take $m \geq n$

2 Bounding Boxes

We define an array A in terms of the numbers

$$A_{i,j} \in \{0, 1\}$$
$$(1 \leq i \leq m, 1 \leq j \leq n)$$

We can then define the bounding box for non-zero A, $B(A)$, as follows

$$B(A) = (1 + \max X - \min X, 1 + \max Y - \min Y)$$

where

$$X = \{i : \exists A_{i,j} = 1\}$$
$$Y = \{j : \exists A_{i,j} = 1\}$$

Essentially, it is the box of the smallest dimensions that can contain all the 1s present in S.

Theorem 1. *Denote by N the number of distinct arrays of size m by n. Let $C(p, q)$ denote the distinct arrays of size p by q and with a bounding box (p, q) (hence we can ignore translational symmetry). Then*

$$N = 1 + \sum_{\substack{p \geq q \geq 1 \\ p+q < m+n}} C(p, q)$$

Proof. We begin by noting that we can partition all of the 2^{mn} arrays into the zero array and the rest in terms of their bounding box and $(\min X, \min Y)$. Since clearly the value of $(\min X, \min Y)$ has a translational symmetry with the same array at $(1,1)$ we can ignore all arrays for which $(\min X, \min Y) \neq (1,1)$ as they are duplicates and hence not distinct. Also clearly since $C(p,q) = C(q,p)$ we only count it in the case that $p \geq q$ (and this way we cover all the arrays since we assumed $m \geq n$). Hence the expression given. □

We denote the set of p by q arrays with bounding box (p,q) as $B(p,q)$. We show a result about $|B(p,q)|$ for $p,q \geq 2$.

Lemma 1. *Let $p,q \geq 2$ and let $x = 2^{p-2}$, $y = 2^{q-2}$ and $z = 2^{(p-2)(q-2)}$. Then*

$$B(p,q) = z((x-1)^2(y-1)^2 + 4xy(x-1)(y-1) + 2x^2y(y-1) + 2x(x-1)y^2 + 7x^2y^2)$$

Proof. Since $p,q \geq 2$ we know there exist four corners of the array. By looking at cases of the corners assuming the values $0,1$ we can find an expression for the total number of array. For example, if all corners are 0, in order for the array to form a bounding box of the correct size we *must* have at least one 1 in the top edge, and in the right edge and the bottom edge and the left edge. There are $(x-1)$ possibilities for the top and bottom edge that fit this criteria, and $(y-1)$ for the left and right edge. Hence since the interior may be filled in any way it has z possibilities and so in this case we get $z((x-1)^2(y-1)^2)$. If we take exactly one corner to be 1, which constitutes four of the sixteen cases by symmetry, we can without loss of generality say the top and left edge can be filled in any way and the right and bottom edge must have at least one to form a bounding box of the correct size. Again we multiply by z and get an additional term $z(4x(x-1)y(y-1))$. If we choose two corners to be one, if the corners share an edge then we either get a $x^2y(y-1)$ or $x(x-1)y^2$ term depending on if it is a vertical or horizontal edge respectively. Hence by the symmetry we get the $z(2x^2y(y-1)+2x(x-1)y^2)$ term. It can be seen that the remaining 7 choices all relax any restrictions on the edges and hence the $z(7x^2y^2)$ term. Hence,

$$B(p,q) = z((x-1)^2(y-1)^2 + 4xy(x-1)(y-1) + 2x^2y(y-1) + 2x(x-1)y^2 + 7x^2y^2)$$

as required.

□

3 Evaluating $C(p,q)$

Since we no longer have to worry about translations we consider the dihedral groups D_4 for $p = q$ and D_2 for $p \neq q$ which cover all symmetries of elements of $B(p,q)$. Since a symmetry on a array of this kind maps onto a array of this kind, we can define group actions $\phi_p : D_4 \times B(p,p) \to B(p,p)$ and $\phi_{pq} : D_2 \times B(p,q) \to B(p,q)$.

Burnside's lemma then gives us the following results for both $p = q$ and $p \neq q$

$$C(p,p) = |B(p,p)/D_4| = \frac{1}{|D_4|} \sum_{d \in D_4} |B(p,p)^d|$$

$$C(p,q) = |B(p,q)/D_2| = \frac{1}{|D_2|} \sum_{d \in D_2} |B(p,q)^d|$$

We look first at the case $p \neq q$ as it is simpler. We consider the number of arrays fixed by each individual symmetry to arrive at the expression

$$C(p,q) = \frac{1}{4}(|B(p,q)| + R(p,q) + R(q,p) + D_R(p,q))$$

Here, R is a reflection in the y-axis and D_R is a rotation by 180 degrees. We now deduce explicit closed forms expressions for R and D_R in the case $p,q \geq 2$ in a similar way to evaluating $|B(p,q)|$

Lemma 2. *Let* $p, q \geq 2$ *and* $x = 2^{\lfloor \frac{p-1}{2} \rfloor}$, $y = 2^{q-2}$, $z = 2^{\lfloor \frac{p-1}{2} \rfloor (q-2)}$. *Then*

$$R(p,q) = z((x-1)^2(y-1) + 2x(x-1)y + x^2 y)$$

Proof. The procedure of the proof is similar to that of Lemma 2. If all corners of the array are 0 we must choose one from the left edge (which then automatically fills the opposite one of the right edge by the symmetry), as well as choosing one from the top and bottom. Because of the symmetry the amount of total choices for the top and bottom edges is x. Considering all the interior choices and taking into account the symmetry (the fact that a choice may fill in an opposing square) we get the term $z((x-1)^2(y-1))$. We cannot take just one corner as the symmetry does not allow it. If we have two corners they are shared by a horizontal edge, hence without loss of generality the top edge and the vertical edges are free of restriction. Since by symmetry there are two of these cases we get the term $2z(x(x-1)y)$. Finally if we have all four corners we get the term $z(x^2 y)$. Hence,

$$R(p,q) = z((x-1)^2(y-1) + 2x(x-1)y + x^2 y)$$

as required. □

Lemma 3. *Let* $p, q \geq 2$ *and* $x = 2^{p-2}$, $y = 2^{q-2}$ *and* $z = 2^{\lfloor \frac{(p-2)(q-2)+1}{2} \rfloor}$. *Then*

$$D_R(p,q) = z((x-1)(y-1) + 3xy)$$

Proof. If there are no corners we must choose one from the top edge and one from the left edge (the bottom and right are then filled due to the symmetry). z is an expression for the number of ways to fill the interior and follows due to the symmetry and the fact that if there is an odd number of points in the interior then there is one which is invariant under the transformation. Therefore we get $z((x-1)(y-1))$. In the other three cases for the corners we have no restrictions on the edges and hence we get $z(3xy)$. Hence

$$D_R(p,q) = z((x-1)(y-1) + 3xy)$$

as required. □

We are now ready to write a closed form expression for $C(p,q)$ in the case that $p \neq q$

Theorem 2. *Let* $p, q \geq 2$ *with* $p \neq q$. *Set* $x = 2^{p-2}$, $y = 2^{q-2}$, $z = 2^{(p-2)(q-2)}$, $\omega = 2^{\lfloor \frac{p-1}{2} \rfloor}$, $\gamma = 2^{\lfloor \frac{q-1}{2} \rfloor}$, $\delta = 2^{(p-2)\lfloor \frac{q-1}{2} \rfloor}$, $\alpha = 2^{\lfloor \frac{p-1}{2} \rfloor (q-2)}$ *and* $\beta = 2^{\lfloor \frac{(p-2)(q-2)+1}{2} \rfloor}$ *Then*

$$\begin{aligned} C(p,q) =& \frac{1}{4}(z((x-1)^2(y-1)^2 + 4xy(x-1)(y-1) + 2x^2y(y-1) + 2x(x-1)y^2 + 7x^2y^2) + \\ & \alpha((\omega-1)^2(y-1) + 2\omega(\omega-1)y + \omega^2 y) + \delta((\gamma-1)^2(x-1) + 2\gamma(\gamma-1)x + \gamma^2 x) + \\ & \beta((x-1)(y-1) + 3xy)) \end{aligned}$$

Proof. The result trivially follows from lemma 2, 3 and 4 and the expression for $C(p,q)$ derived via Burnside's lemma. □

We now look at the case where $p = q$. Introducing the new notation Rot and Diag for rotation by 90 degrees anticlockwise and reflections in the diagonal, we arrive at the expression

$$C(p,p) = \frac{1}{8}(|B(p,p)| + 2R(p,p) + D_{rot}(p,p) + 2\text{Rot}(p) + 2\text{Diag}(p))$$

We first look at evaluating $\text{Rot}(p)$ for $p \geq 2$. We start by considering the number of rings in an array, where a ring is formed by taking a point and applying the rotation four times. Clearly since each ring is of size four, except for possibly the central invariant square in the case of odd p, we have a total number of rings $r = \lceil \frac{p^2}{4} \rceil$. Since we have $p-1$ rings in the outer shell (the edge and corners) we must have $r - p - 1$ rings in the interior.

Lemma 4. *Let* $p \geq 2$. *Set* $r = \lceil \frac{p^2}{4} \rceil$, $x = 2^{p-1}$, $y = 2^{r-p+1}$ *Then*

$$Rot(p) = (x-1)y$$

Proof. We must select at least one ring from the outer ring (to fulfill the bounding box criteria) hence the $x-1$. Since the interior can be set in y ways we conclude that $\mathrm{Rot}(p) = (x-1)y$ $\square$

Finally, we evaluate $\mathrm{Diag}(p)$

Lemma 5. *Let $p \geq 2$. Set $x = 2^{p-2}$ and $y = 2^{\frac{(p-1)(p-2)}{2}}$. Then*

$$Diag(p) = y((x-1)^2 + 5x^2 + 2x(x-1))$$

Proof. Without loss of generality consider the diagonal going from bottom left to top right. If we take the top left corner, by the symmetry we must also take the bottom right and hence we are free of bounding box restrictions, giving the term $4z(x^2)$ as z denotes all the ways to take the interior with respect to the symmetry (it is a triangular number) and there are four ways to choose the other two corners and there are x ways to choose both the top/right and bottom/left edge pairs. If we don't choose the top left corner we have the case where we don't choose any corners, $z((x-1)^2)$, the case where we choose one of the corners on the diagonal, $2z(x(x-1))$, and the case where we choose both on the diagonal, $z(x^2)$. Hence,

$$\mathrm{Diag}(p) = y((x-1)^2 + 5x^2 + 2x(x-1))$$

as required. $\square$

We are now ready to write a closed form expression for $C(p,q)$ in the case that $p = q$

Theorem 3. *Let $p \geq 2$. Set $x = 2^{p-2}$, $y = 2^{(p-2)^2}$, $z = 2^{\lfloor \frac{p-1}{2} \rfloor}$, $\omega = 2^{(p-2)\lfloor \frac{p-1}{2} \rfloor}$, $\delta = 2^{\lfloor \frac{(p-2)^2+1}{2} \rfloor}$, $\gamma = 2^{\frac{(p-1)(p-2)}{2}}$, $r = \lceil \frac{p^2}{4} \rceil$, $\alpha = 2^{p-1}$ and $\beta = 2^{r-p+1}$. Then*

$$\begin{aligned} C(p,p) =& \frac{1}{8}(y((x-1)^4 + 4x^2(x-1)^2 + 4x^3(x-1) + 7x^4) + 2\omega((z-1)^2(x-1) + 2z(z-1)x + z^2x) + \\ & \delta((x-1)^2 + 3x^2) + 2(\alpha-1)\beta + 2\gamma((x-1)^2 + 5x^2 + 2x(x-1))) \end{aligned}$$

Proof. The result follows trivially from lemmas 1,2,3,4,5 and from using Burnside's lemma to obtain an expression for $C(p,p)$ in terms of fixed elements with respect to its symmetry group. $\square$

With closed form expressions for $C(p,q)$ with $p,q \geq 2$ we must now calculate $C(p,1)$ for $p \geq 1$. $C(1,1) = 1$ trivially.

Theorem 4. *Let $p \geq 2$. Then*

$$C(p,1) = \frac{1}{4}(2^{p-1} + 2^{1+\lfloor \frac{p-1}{2} \rfloor})$$

Proof. We must have both ends filled to fulfill the bounding box criteria. Hence under identity there are 2^{p-2} fixed elements. Reflection in the x-axis is identity in this case, so we have two identities giving 2^{p-1}. In this case double rotation is equivalent to reflection in the y-axis, of which there are $2^{\lfloor \frac{p-1}{2} \rfloor}$ ways to fill the shape. Since there are two of these as well, we arrive at the result via Burnside's. $\square$

4 Results

Using Theorem 1 and our knowledge of closed form expressions for $C(p,q)$ we can compute a table of values of N for m by n arrays. Here is a table for $m = n$, $1 \geq n \geq 9$

1	2
2	6
3	86
4	7626
5	3956996
6	8326366368
7	69277957195904
8	2287898999182608384
9	301053169143557925109760