Computational Characterization of a(n) = A038547(2^n)

Hartmut F. W. Hoft, 2023-02-07

A list of properties for the prime factorization $\prod_{i=1}^{s} p_{i+1}^{2^{f_i}-1}$ of certain numbers with 2^n odd divisors:

(1) $3 = p_2 < \ldots < p_{s+1}$ are the first consecutive s odd primes

(2) $f_1 \geq \ldots \geq f_s \geq 1$ and $\sum_{i=1}^{s} f_i = n$

(3) $p_{s+1} < {p_{i+1}}^{2^{f_i}}$, for all $1 \leq i < s$

Theorem: For every $n \geq 1$, equivalent are

(a) a(n) is the least number with 2^n odd divisors; a(n) = A038547(2^n).

(b) a(n) = $\prod_{i=1}^{s} p_{i+1}^{2^{f_i}-1}$ satisfies properties (1) - (3) and a(n-1) | a(n).

Proof by induction on n:

(a) ⇒ (b): Properties (1) - (3) are obviously necessary.

Let a(n-1) = $\prod_{i=1}^{r} p_{i+1}^{2^{e_i}-1}$ and a(n) = $\prod_{i=1}^{s} p_{i+1}^{2^{f_i}-1}$ then b = $\prod_{i=1}^{s-1} p_{i+1}^{2^{f_i}-1} \times p_{s+1}^{2^{f_s-1}-1}$ has 2^{n-1} odd divisors so that a(n-1) ≤ b < a(n).

If r > s and $1 \leq k \leq s$ is the smallest index with $e_k < f_k$ then $p_{r+1} < p_{k+1}^{2^{e_k}} \leq p_{k+1}^{2^{f_k-1}}$ by induction so that $\frac{a(n)}{p_{k+1}^{2^{f_k-1}}} \times p_{r+1}$ < a(n) has 2^n odd divisors, contradicting minimality of a(n).

If a(n-1) ∤ a(n) then every prime dividing a(n-1) divides also a(n) since r ≤ s. Therefore, there is k ≤ r such that $p_{k+1}^{2^{e_k}-1}$ | a(n-1) and $p_{k+1}^{2^{f_k}-1}$ | a(n) and $e_k > f_k$.

If r < s then $p_{r+1} < p_{s+1} < p_{k+1}^{2^{f_k}} \leq p_{k+1}^{2^{e_k-1}}$ and $\left(\prod_{i=1,\, i\neq k}^{r} p_{i+1}^{2^{e_i}-1}\right) \times p_{k+1}^{2^{e_k-1}-1} \times p_{s+1}$ < a(n-1) and has 2^{n-1} odd divisors, contradicting minimality of a(n-1).

If r = s then there are k, m ≤ r with $e_k > f_k$ and $f_m > e_m$ since $1 + \sum_{i=1}^{s} e_i = \sum_{i=1}^{s} f_i$.

If k < m then $e_k > f_k \geq f_m > e_m$ so that $p_{k+1}^{2^{f_m-1}} \times p_{m+1}^{-2^{f_k}} < 1$ and $\left(\prod_{i=1,\, i\neq k,m}^{s} p_{i+1}^{2^{f_i}-1}\right) \times p_{k+1}^{2^{f_m-1}-1} \times p_{m+1}^{2^{f_k+1}-1}$ < a(n) has 2^n odd divisors, contradicting minimality of a(n).

If k > m then $f_m > e_m \geq e_k > f_k$ so that $p_{m+1}^{2^{f_k-1}} \times p_{k+1}^{-2^{f_m}} < 1$ and $\left(\prod_{i=1,\, i\neq m,k}^{s} p_{i+1}^{2^{f_i}-1}\right) \times p_{m+1}^{2^{f_k-1}-1} \times p_{k+1}^{2^{f_m+1}-1}$ < a(n) has 2^n odd divisors, contradicting minimality of a(n).

(b) ⇒ (a): Assume that $a(n+1) = \prod_{i=1}^{t} {p_{i+1}}^{2^{g_i}-1}$ satisfies properties (1) - (3), a(n) | a(n+1) and $a(n) = \prod_{i=1}^{s} {p_{i+1}}^{2^{f_i}-1}$ is the least number with 2^n odd divisors and thus satisfies (b) by induction. Let ${p_{k+1}}^{2^{f_k}} = \min({p_{i+1}}^{2^{f_i}} \mid 1 \le i \le s)$ then

$$a(n+1) = \begin{cases} a(n) \times p_{s+2} & \text{if } p_{s+2} < {p_{k+1}}^{2^{f_k}} \\ a(n) \times {p_{k+1}}^{2^{f_k}} & \text{otherwise} \end{cases}$$ is the smallest number with 2^{n+1} odd divisors.

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The efficiency of computing a(n) can be improved significantly by testing the values of the primaries of a(n) only at those indices where a run of identical numbers in the partition of n starts. For example, a(24) would require only 2 instead of 21 computations for determining the minimum of its primaries.