Notes on A356993

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Let A(n) = A356933(n). The definition is

$$A(n) = a(n - a(n - a(n - a(n)))), \text{for } n \geq 2, \tag{1}$$

where $a(n) = \text{A356988}(n)$ is defined by the nested recurrence

$$a(n) = n - a(a(n - a(a(a(n-1))))) \tag{2}$$

with the initial condition a(1) = 1.

The sequence {a(n)} is slow [1], that is, $a(n+1) - a(n) \in \{0, 1\}$ for $n \geq 1$. In general, if the sequences $\{u(n) : n \geq 1\}$ and $\{v(n) : n \geq 1\}$ are both slow then it is easy to see that (i) the sequence $\{n - u(n)\}$ is slow and (ii) the composition sequence $\{v(u(n))\}$ is slow. It therefore follows from the definition (1) that the sequence $\{A(n)\}$ is slow.

In order to analyse the structure of the sequence $\{A(n)\}$ we will need following facts about A356988.

The terms of A356988 are completely determined by the following two results [1]:

a) for $n >= 2$

$$a(\mathrm{L}(n-1) + j) = \mathrm{F}(n) \tag{3}$$

for $0 <= j <= \mathrm{F}(n-3)$, where $\mathrm{F}(n) = \text{A000045}(n)$, the n-th Fibonacci number with $\mathrm{F}(-1) = 1$ and $\mathrm{L}(n) = \text{A000032}(n)$, the n-th Lucas number (recall that $\mathrm{L}(n) = \mathrm{F}(n+1) + \mathrm{F}(n-1)$).

b) for $n >= 2$,

$$a(\mathrm{F}(n+1) + j) = \mathrm{F}(n) + j \tag{4}$$

for $0 <= j <= \mathrm{F}(n-1)$.

In addition, we will require the following results, which are easy consequences of (3) and (4):

$$\mathrm{a}(2\mathrm{F}(k)) = \mathrm{L}(k-1) \text{ for } k \geq 2 \tag{5}$$

$$\mathrm{a}(3\mathrm{F}(k)) = 2\mathrm{F}(k) \text{ for } k \geq 1 \tag{6}$$

$$\mathrm{a}(4\mathrm{F}(k)) = \mathrm{F}(k-2) \text{ for } k \geq 2 \tag{7}$$

$$\mathrm{a}(4\mathrm{F}(k) + F(k-1)) = 3\mathrm{F}(k) \text{ for } k \geq 1 \tag{8}$$

For example, to prove (8) we note that $4\mathrm{F}(k) + \mathrm{F}(k-1) = \mathrm{F}(k+3) + \mathrm{F}(k-2)$ and hence $a\,(4\mathrm{F}(k) + \mathrm{F}(k-1)) = a\,(\mathrm{F}(k+3) + \mathrm{F}(k-2)) = \mathrm{F}(k+2) + \mathrm{F}(k-2)$ (by (4)) $= 3F(k)$.

The structure of $\{A(n)\}$.

The line graph of the sequence $\{A(n)\}$ consists of a series of plateaus (where the value of the ordinate $A(n)$ remains constant as n increases) joined by lines of slope 1. The heights of the plateaus are alternately Fibonacci numbers and Lucas numbers. More precisely, we have

1) $\{A(n)\}$ has the constant value $\mathrm{F}(k)$ on the integer interval $[3\mathrm{F}(k), \mathrm{L}(k+1)]$ for $k \geq 3$.

Proof. It suffices to show that the sequence takes on the same value $\mathrm{F}(k)$ at both endpoints of the interval, that is, $A(3\mathrm{F}(k)) = \mathrm{F}(k) = A(\mathrm{L}(k+1))$. Then, since the sequence $\{A(n)\}$ is slow, it follows that it must have the constant value $\mathrm{F}(k)$ throughout the integer interval $[3\mathrm{F}(k), \mathrm{L}(k+1)]$.

For the left endpoint of the interval we calculate

$$\begin{aligned}
A\,(3\mathrm{F}(k)) &= a\,(3\mathrm{F}(k) - a\,(3\mathrm{F}(k) - a\,(3\mathrm{F}(k) - a\,(3\mathrm{F}(k))))) \\
&= a\,(3\mathrm{F}(k) - a\,(3\mathrm{F}(k) - a\,(3\mathrm{F}(k) - 2\mathrm{F}(k)))) \text{ by (6)} \\
&= a\,(3\mathrm{F}(k) - a\,(3\mathrm{F}(k) - \mathrm{F}(k-1))) \text{ by (4)} \\
&= a\,(3F(k)) - a\,(\mathrm{L}(k) + \mathrm{F}(k-4)) \\
&= a\,(3\mathrm{F}(k) - \mathrm{F}(k+1)) \text{ by (3)} \\
&= a\,(\mathrm{L}(k-1)) \\
&= \mathrm{F}(k) \text{ by (3)}
\end{aligned}$$

where we made use of the easily proved identities

$$3\mathrm{F}(k) - \mathrm{F}(k-1) = \mathrm{L}(k) + \mathrm{F}(k-4) \text{ and } 3\mathrm{F}(k) - \mathrm{F}(k+1) = \mathrm{L}(k-1).$$

For the right endpoint of the interval we calculate

$$\begin{aligned}
A\,(\mathrm{L}(k+1)) &= a\,(\mathrm{L}(k+1) - a\,(\mathrm{L}(k+1) - a\,(\mathrm{L}(k+1) - a\,(\mathrm{L}(k+1))))) \\
&= a\,(\mathrm{L}(k+1) - a\,(\mathrm{L}(k+1) - a\,(\mathrm{L}(k+1) - \mathrm{F}(k+2)))) \text{ by (3)} \\
&= a\,(\mathrm{L}(k+1) - a\,(\mathrm{L}(k+1) - a\,(\mathrm{F}(k)))) \\
&= a\,(\mathrm{L}(k+1)) - a\,(\mathrm{L}(k+1) - \mathrm{F}(k-1)) \text{ by (4)} \\
&= a\,(\mathrm{L}(k+1) - a\,(3\mathrm{F}(k))) \\
&= a\,(\mathrm{L}(k+1) - 2\mathrm{F}(k)) \text{ by (6)} \\
&= a\,(\mathrm{F}(k+1)) \\
&= \mathrm{F}(k) \text{ by (4)}
\end{aligned}$$

where we made use of the easily proved identities

$$\mathrm{L}(k+1) = \mathrm{F}(k+2) + \mathrm{F}(k),\ \mathrm{L}(k+1) - \mathrm{F}(k-1) = 3\mathrm{F}(k) \text{ and } \mathrm{L}(k+1) - 2\mathrm{F}(k) = \mathrm{F}(k+1).$$

□

2) $\{A(n)\}$ has the constant value $\mathrm{L}(k-1)$ on the integer interval $[4\mathrm{F}(k), 4\mathrm{F}(k) + \mathrm{F}(k-1)]$ for $k \geq 2$.

Proof. Again, by the slowness property of the sequence, it suffices to show that the sequence takes on the same value $\mathrm{L}(k-1)$ at both endpoints of the interval:

$$A(4\mathrm{F}(k)) = \mathrm{L}(k-1) = A(4\mathrm{F}(k) + \mathrm{F}(k-1)).$$

For the left endpoint of the interval we calculate

$$\begin{aligned} A\,(4\mathrm{F}(k)) &= a\,(4\mathrm{F}(k) - a\,(4\mathrm{F}(k) - a\,(4\mathrm{F}(k) - a\,(4\mathrm{F}(k))))) \\ &= a\,(4\mathrm{F}(k) - a\,(4\mathrm{F}(k) - a\,(4\mathrm{F}(k) - \mathrm{F}(k+2)))) \text{ by } (7) \\ &= a\,(4\mathrm{F}(k) - a\,(4\mathrm{F}(k) - a\,(\mathrm{L}(k-1)))) \\ &= a\,(4F(k)) - a\,(4\mathrm{F}(k) - \mathrm{F}(k)) \text{ by } (3) \\ &= a\,(4\mathrm{F}(k) - 2\mathrm{F}(k)) \text{ by } (6) \\ &= \mathrm{L}(k-1) \text{ by } (5) \end{aligned}$$

where we made use of the easily proved identity

$$4\mathrm{F}(k) - \mathrm{F}(k+2) = \mathrm{L}(k-1).$$

For the right endpoint of the interval we set $N = 4\mathrm{F}(k) + \mathrm{F}(k-1)$ and calculate

$$\begin{aligned} A\,(4\mathrm{F}(k) + \mathrm{F}(k-1)) &= a\,(N - a\,(N - a\,(N - a\,(N)))) \\ &= a\,(N - a\,(N - a\,(4\mathrm{F}(k) + \mathrm{F}(k-1) - 3\mathrm{F}(k)))) \text{ by } (8) \\ &= a\,(N - a\,(N - a\,(\mathrm{F}(k+1)))) \\ &= a\,(N - a\,(4\mathrm{F}(k) + \mathrm{F}(k-1) - \mathrm{F}(k))) \text{ by } (4) \\ &= a\,(4\mathrm{F}(k) + \mathrm{F}(k-1) - a\,(\mathrm{L}(k+1))) \\ &= a\,(4\mathrm{F}(k) + \mathrm{F}(k-1) - \mathrm{F}(k+2)) \text{ by } (3) \\ &= a\,(2\mathrm{F}(k)) \\ &= \mathrm{L}(k-1) \text{ by } (5) \end{aligned}$$

where we made use of the easily proved identities

$$4\mathrm{F}(k) + \mathrm{F}(k-1) - 3\mathrm{F}(k) = \mathrm{F}(k+1),\ 4\mathrm{F}(k) + \mathrm{F}(k-1) - \mathrm{F}(k) = \mathrm{L}(k+1)$$

and

$$4\mathrm{F}(k) + \mathrm{F}(k-1) - \mathrm{F}(k+2) = 2\mathrm{F}(k).$$

□

3) Next we show that on the interval $[\mathrm{L}(k+1), 4\mathrm{F}(k)]$ between the intervals $[3\mathrm{F}(k), \mathrm{L}(k+1)]$ and $[4\mathrm{F}(k), 4\mathrm{F}(k)+\mathrm{F}(k-1)]$ the line graph of the sequence $\{A(n)\}$ has constant slope 1. This is an immediate consequence of the slowness of $\{A(n)\}$ since on the integer interval $[\mathrm{L}(k+1), 4\mathrm{F}(k)]$ of length $\mathrm{F}(k-2)$ the sequence increases in value by $A\left(4\mathrm{F}(k)\right) - A\left(\mathrm{L}(k+1)\right) = \mathrm{L}(k-1) - \mathrm{F}(k) = \ \mathrm{F}(k-2)$.

4) By an exactly similar calculation we find the slope of line graph of the sequence is also 1 on the interval from the end of one plateau at abscissa $n = 4\mathrm{F}(k) + \mathrm{F}(k-1))$ to the start of the next plateau at abscissa $n = 3\mathrm{F}(k+1)$.

References

[1] Peter Bala, Notes on A356988