Derivation of the recurrence in A355881

Each column can be colored with two types of patterns: A(aba) or B(abc) where a,b,c are distinct colors. Any pattern of type A can be followed by one of p(k) A-patterns or one of q(k) B-patterns. Any pattern of type B can be followed by one of r(k) A-patterns or one of s(k) B-patterns. Let A(k,n) and B(k,n) be the number of ways to color the grid such that the n-th column has the corresponding pattern aba or abc:

$$\mathrm{A(k,n)} = \mathrm{p(k)} \cdot \mathrm{A(k,n-1)} + \mathrm{q(k)} \cdot \mathrm{B(k,n-1)} \text{ with } \mathrm{A(k,1)} = 1,$$
$$\mathrm{B(k,n)} = \mathrm{r(k)} \cdot \mathrm{A(k,n-1)} + \mathrm{s(k)} \cdot \mathrm{B(k,n-1)} \text{ with } \mathrm{B(k,1)} = \mathrm{k},$$

With this first recurrence, a second one can be generated for
$T(k,n) = A(k,n) + B(k,n)$:
$T(k,n) = u(k) \cdot T(k,n-1) + v(k) \cdot T(k,n-2)$ with
$u(k) = p(k) + s(k)$ and $v(k) = q(k) \cdot r(k) - p(k) \cdot s(k)$, see annotation.

$T(k,1) = k + 1 \; and \; T(k,2) = p(k) + k \cdot q(k) + r(k) + k \cdot s(k)$
with $p(k) = k^2 + k + 1, q(k) = k \cdot (k^2 + 1)$,
$r(k) = k\text{^}2 + 1 \; and \; s(k) = k\text{^}3 + 2k - 1$ (derivations below).

Simplified formula:
$T(k,n) = k(k^2 + k + 3) \cdot T(k,n-1) - (k^4 + k^3 + k^2 - 1) \cdot T(k,n-2)$
with $T(k,1) = k + 1, T(k,2) = (k^2 + k + 1)^2$

Annotation:
Set $x = A(k,n-2), y = B(k,n-2)$. Then, referring to the first recurrence, solve this system of equations:

$$T(k,n) = u(k) \cdot T(k,n-1) + v(k) \cdot T(k,n-2)$$
$$T(k,n+1) = u(k) \cdot T(k,n) + v(k) \cdot T(k,n-1)$$

The solution $(u(k), v(k))$ does not depend on x, y. That is a confirmation of the second recurrence.

Derivation of $p(k), q(k), r(k) \; and \; s(k)$:
$k + 2$ colors: $1,2,\ldots,k+2$. Given pattern of a column: 121 or 123; colors in the next column: xyz with $1 <= x,y,z <= k+2$.
Restrictions: $x \neq 1; y \neq 2, x; \; z \neq y$.
Moreover, $z \neq 1$ for 121and $z \neq 3$ for 123.

$121 \; following \; by \; xyx$		$123 \; following \; by \; xyx$	
$x = 2$	$x > 2$	$x = 2$	$x > 3$
$y \neq 2$	$y \neq 2, x$	$y \neq 2$	$y \neq 2, x$
$k + 1 \quad +$	$k \cdot k$	$k + 1 \quad +$	$(k-1) \cdot k$
$p(k) = k^2 + k + 1$		$r(k) = k^2 + 1$	

$121\ followed\ by\ xyz\ with\ z \neq x$			
$x = 2$	$x = 2$	$x > 2$	$x > 2$
$y = 1$	$y \neq 1,2$	$y = 1$	$y \neq 1,2,x$
$z \neq 1,2$	$z \neq 2,3,y$	$z \neq 1,x$	$z \neq 1,x,y$
$k \quad + \ k \cdot (k-1) \quad + \ k \cdot k \quad + \ k \cdot (k-1)^2$			
$q(k) = k(k^2+1)$			

$123\ followed\ by\ xyz\ with\ z \neq x$				
$x = 2$	$x = 2$	$x = 3$	$x > 3$	$x > 3$
$y = 3$	$y \neq 2,3$	$y \neq 2,3$	$y = 3$	$y \neq 2,3,x$
$z \neq 2,3$	$z \neq 2,3,y$	$z \neq 2,3$	$z \neq 3,x$	$z \neq 3,x,y$
$k \quad + \ k \cdot (k-1) \quad + \ k \cdot k \quad + \ (k-1) \cdot k \quad + \ (k-1)^3$				
$s(k) = k^3 + 2k - 1$				