Test of the k-tuple conjecture on triples, quadruples and quintuples of twin primes (represented by their average).

Twin primes x such that x is the first and x+d the last of m successive twins.

1. Triples of twin primes with m=3 and d=18 (A350541)

There are two triples $(x, x+h, x+18)$: $h = 6\ or\ h = 12$

Example: $(x, x+6, x+18)$ corresponds to a 6-tupel of primes:

$$(p, p+2, p+6, p+8, p+18, p+20) \text{ with } p = x-1.$$

From $(p+2\cdot 0, p+2\cdot 1, p+2\cdot 3, p+2\cdot 4, p+2\cdot 9, p+2\cdot 10)$, we keep the 6-tuple $\overrightarrow{v_1} = (0,1,3,4,9,10)$. For any odd prime q, $w(q,\overrightarrow{v_1})$ is the number of distinct residues of $\overrightarrow{v_1}\ (mod\ q)$. For q=3, the residues are 0,1,0,1,0,1 with $w(3,\overrightarrow{v_1}) = 2$. The same way, we find $w(5,\overrightarrow{v_1}) = 4$, $w(7,\overrightarrow{v_1}) = 5$ and $w(q,\overrightarrow{v_1}) = 6$ for q>10.

For the number $N_1(y)$ of prime 6-tuples $(p, p+2, p+6, p+8, p+18, p+20)$, $p < y$, the conjecture is

$N_1(y) \sim C(\overrightarrow{v_1}) \int_2^y \frac{dt}{ln^6 t}$ with $C(\overrightarrow{v_1}) = 2^5 \prod_q \frac{1-\frac{w(q,\overrightarrow{v_1})}{q}}{\left(1-\frac{1}{q}\right)^6}$ over all odd primes q.

The result is $C(\overrightarrow{v_1}) = 34.5973.. \approx 34.6$.

The other triple of twin primes $(x, x+12, x+18)$ with $\overrightarrow{v_2} = (0,1,6,7,9,10)$ yields the same result so that, for both types of triples as a whole, the constant is $C = 2C_1 \approx 69.2$.

Table with $L(6,y) = \int_2^y \frac{dt}{ln^6 t}$:

y	$N(y)$	$L(6,y)$	$N(y)/L(6,y) \approx C$
$2\cdot 10^{10}$	10750	157.17	68.39723393
$4\cdot 10^{10}$	17878	259.31	68.94536828
$6\cdot 10^{10}$	24098	348.98	69.05229524
$8\cdot 10^{10}$	29853	431.52	69.18183298
$1\cdot 10^{11}$	35314	509.25	69.34510228

The lower integration limit is a=2 by convention. A different choice would, asymptotically, not affect the constancy of values in the last column. But as the conjecture is tested for $y < 10^{11}$, it may be necessary to adjust the limit:

r	$\int_2^3 \frac{dt}{ln^r t}$	$\int_3^y \frac{dt}{ln^r t}$
6	2.542	506.5
8	3.944	1.169
10	6.430	0.168

For r=6, the first integral is small compared with the second. a=2 or a=3 does not make a great difference. For r=8, a=6.9 is a good choice and for r=10, even the best choice a=70 is not very good, see quintuples.

2. Quadruples of twin primes with m=4 and d=30 (A350542)

There are 6 quadruples $(x, x+6k_1, x+6k_2, x+30)$ with $1 \le k_1 < k_2 \le 4$. Three of them are excluded for divisibility reasons. Remaining quadruples:
1) $(x, x+6, x+18, x+30)$, 2) $(x, x+12, x+24, x+30)$, 3) $(x, x+12, x+18, x+30)$
Example: $\overrightarrow{v_1} = (0,1,3,4,9,10,15,16)$ leads to:

$$N_1(y) \sim C(\overrightarrow{v_1}) \int_a^y \frac{dt}{ln^8 t} \text{ with } C(\overrightarrow{v_1}) = 2^7 \prod_q \frac{1-\frac{w(q,\overrightarrow{v_1})}{q}}{\left(1-\frac{1}{q}\right)^8} \text{ over all odd primes q.}$$

For the three quadruples of twin primes, we find: $C(\overrightarrow{v_1}) = C(\overrightarrow{v_2}) = 475.4$, $C(\overrightarrow{v_3}) = 297.1$ and $C = C(\overrightarrow{v_1}) + C(\overrightarrow{v_2}) + C(\overrightarrow{v_3}) \approx 1248$.
Table with $L(8,y) = \int_a^y \frac{dt}{ln^8 t}$ and a=6.9:

y	$N(y)$	$L(8,y)$	$N(y)/L(8,y) \approx C$
$2 \cdot 10^{10}$	393	0.33022	1190
$4 \cdot 10^{10}$	623	0.50609	1231
$6 \cdot 10^{10}$	805	0.65396	1231
$8 \cdot 10^{10}$	985	0.78642	1253
$1 \cdot 10^{11}$	1134	0.90861	1248

The parameter $a = 6.9$ was chosen such that $\frac{N(y)}{L(8,y)} = C$ for $y = 10^{11}$. Then the values in the last column are fairly constant.

3. Quintuples of twin primes with m=5 and d=48 (A350543)

A priori, we have to consider all quintuples of even numbers $(x, x+6k_1, x+6k_2, x+6k_3, x+48)$ with $1 \le k_1 < k_2 < k_3 \le 7$.
Their number is $\binom{7}{3} = 35$. Each quintuple corresponds to a 10-tuple of odd numbers: $(x \pm 1, x+6k_1 \pm 1, x+6k_2 \pm 1, x+6k_3 \pm 1, x+48 \pm 1)$.
The differences between these numbers and $x-1$, divided by 2, are

$$\overrightarrow{v_j} = (0,1,3k_1,3k_1+1,3k_2,3k_2+1,3k_3,3k_3+1,24,25), j = 1..35$$

with the constant $C(\overrightarrow{v_j}) = 2^9 \prod_q \frac{1-\frac{w(q,\overrightarrow{v_j})}{q}}{\left(1-\frac{1}{q}\right)^{10}}$ over all odd primes q where $w(q,\overrightarrow{v_j})$ is the number of distinct residues of $\overrightarrow{v_j}\ (mod\ q)$. 27 constants vanish, eight remain. Summation: $C = \sum_{j=1}^{8} C(\overrightarrow{v_j}) = 18606$. $a = 70$ is chosen such that $\frac{N(y)}{L(10,y)} \approx C$ for $y = 10^{11}$. This leads to the table:

y	$N(y)$	$L(10,y)$	$N(y)/L(10,y)$
$2 \cdot 10^{10}$	13	0.000709	18333
$4 \cdot 10^{10}$	23	0.001014	22684
$6 \cdot 10^{10}$	24	0.001259	19063
$8 \cdot 10^{10}$	27	0.001473	18335
$1 \cdot 10^{11}$	31	0.001666	18610

The values in the last column differ quite a lot. For a better constancy, y should be much greater than 10^{11}.

The formalism of the conjecture allows us a second test which does not depend on the parameter a: We can evaluate the expected frequencies of the eight different types of quintuples relative to the frequency of all quintuples. The constants $C(\vec{v_j})$ only differ by the factor $c_j = \prod_q (q - w(q, \vec{v_j}))$ over all odd primes $q < 25$ (last term of $\vec{v_j}$). Then the expected relative frequency of the corresponding quintuple of twin primes is $f_j = \frac{c_j}{\sum_{i=1}^{8} c_i}$ (last column):

Form of the quintuples: (x, x+a, x+b, x+c, x+d)	Relative frequencies	
(a,b,c,d)	observed	expected
6,18,30,48	11/31=0.355	0.237
6,30,36,48	5/31=0.161	0.150
6,18,36,48	3/31=0.097	0.075
18,30,42,48	6/31=0.194	0.237
12,18,42,48	3/31=0.097	0.150
12,30,42,48	2/31=0.065	0.038
12,18,30,48	1/31=0.032	0.075
18,30,42,48	0	0.038

The observed and expected frequencies differ quite a lot. The number 31 of observed terms is too small for a better match.