A note on spanning trees

Yoshiaki Doi[*], Tsuyoshi Miezaki[†], Tomoki Nakamigawa[‡], Tadashi Sakuma[§], Etsuo Segawa[¶], Hidehiro Shinohara[‖], Shunya Tamura[**], Yuho Tanaka[††], Kosuke Toyota[‡‡]

Abstract

In this short note, we prove a conjecture posed by Professor Simon Plouffe.

1 Introduction

Let $\tau_3(n)$ be the number of spanning trees in the 3rd power of a cycle of length n. In [1], Professor Simon Plouffe stated the following conjecture:

[*]Faculty of Education, Art and Science, Yamagata University, Yamagata 990–8560, Japan

[†]Faculty of Education, Ryukyu University, Okinawa, 903–0213, Japan, miezaki@edu.u-ryukyu.ac.jp

[‡]Department of Information Science Shonan Institute of Technology 1-1-25 Tsujido-Nishikaigan, Fujisawa 251–8511, Japan

[§]Faculty of Science, Yamagata University, Yamagata 990–8560, Japan

[¶]Graduate School of Education Center and Graduate School of Environment Information Sciences, Yokohama National University, Hodogaya, Yokohama 240–8501, Japan.

[‖]Institute for Excellence in Higher Education, Tohoku University, Miyagi 980–8576, Japan

[**]Faculty of Education, Art and Science, Yamagata University, Yamagata 990–8560, Japan

[††]Faculty of Education, Art and Science, Yamagata University, Yamagata 990–8560, Japan

[‡‡]Faculty of Education, Art and Science, Yamagata University, Yamagata 990–8560, Japan

Conjecture 1.1.

$$\begin{cases} \tau_3(n) = 2nT(n)^2 \text{ if } n \text{ is even,} \\ \tau_3(n) = nT(n)^2 \text{ if } n \text{ is odd.} \end{cases}$$

where

$$T(n+8) = 4T(n+6) + T(n+4) + 4T(n+2) - T(n).$$

We remark that the original conjecture in [1] provided ambiguous information, which is already fixed in [1].

In this short note, we give a proof of Conjecture 1.1. The proof is a consiquence of [2, Theorem 1].

This paper is organized as follows. In Section 2, we give a proof of Conjecture 1.1. In Section 3, using a result of [2], we give an anothe expression for $\tau_3(n)$.

2 Proof of Conjecture 1.1

Let $T(n)$ be the numbers in A005822. Then we have

Theorem 2.1 ([3])**.**

$$\begin{cases} \tau_3(n) = 2nT(n)^2 \text{ if } n \text{ is even,} \\ \tau_3(n) = nT(n)^2 \text{ if } n \text{ is odd.} \end{cases}$$

where

$$T(n+8) = 4T(n+6) + T(n+4) + 4T(n+2) - T(n).$$

Proof. The proof is similar to the discussion in [3, p.347 Theorem 9]. Let

$$f = 1 + 3x + 6x^2 + 3x^3 + x^4.$$

We denote by

$$a_1, a_2$$

its roots up to conjugate. Let

$$a(n) := \frac{(1-a_1^n)(1-a_2^n)}{\sqrt{14}\sqrt{(a_1a_2)^n}}$$

Then

$$\tau_3(n) = na(n)^2$$

and we have $a(n)$

$$a(n+4) = \sqrt{2}a(n+3) + a(n+2) + \sqrt{2}a(n+1) - a(n)$$

Then we obtain the following:

$$a(n+8) = 4a(n+6) + a(n+4) + 4a(n+2) - a(n).$$

It is easy to check that

$$\begin{cases} T(n) = 1/\sqrt{2}a(n) \text{ if } n \text{ is even,} \\ T(n) = a(n) \text{ if } n \text{ is odd.} \end{cases}$$

□

3 An expression for $\tau_3(n)$

By [2, Theorem 1], we have the following:

Theorem 3.1 ([2, Theorem 1])**.** *Let*

$$T(n,z) := \cos(n \arccos(z))$$

$$z_1 := \frac{-3+\sqrt{-7}}{4}, z_2 := \frac{-3-\sqrt{-7}}{4}.$$

Then we have the following:

$$\tau_3(n) := \frac{2n}{7}(T(n,z_1)-1)(T(n,z_2)-1).$$

Acknowledgments

The authors are supported by JSPS KAKENHI (18K03217, 18K03388).

References

[1] OEIS Foundation Inc. (2020), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A005822.

[2] A. D. Mednykh, I. A. Mednykh, The number of spanning trees in circulant graphs, its arithmetic properties and asymptotic, *Discrete Math.* **342** (2019), no. 6, 1772–1781.

[3] Y. Zhang, X. Yong, M. J. Golin, *The number of spanning trees in circulant graphs*, *Discrete Math.*, **223** (2000), no. 1–3, 337–350.