The Hamming weight $w(n)$ is the number of 1s in n when written in binary. Is there some effective bound on Fibonacci numbers F_n with $w(F_n) \leq x$ for a given x?

Since you specify "effective" in the question I guess you know this already, but just in case: there are only finitely many such n, because $2^{e_1} + \cdots + 2^{e_x} = (\varphi^n - \varphi^{-n})/\sqrt{5}$ is an S-unit equation in $x+2$ variables over $\mathbf{Q}(\sqrt{5})$; but in general no effective proof is known for such a result (though the *number* of solutions of $w(F_n) \leq x$ may be effectively bounded). – Noam D. Elkies Mar 2 '14 at 6:28

The case $x = 2$ is still tractable. If $F_n = 2^e + 2^f$ with $e < f$ then $e < 5$, else $F_n \equiv 0 \bmod 2^5$, which happens **iff** $n \equiv 0 \bmod 24$, and then $7 \mid 21 = F_8 \mid F_{24} \mid F_n$, which is impossible because $2^e + 2^f$ is never a multiple of 7. So we have only a few candidates for e, and we can deal with each of them separately, possibly even by elementary means, to show that $(n, e, f) = (12, 4, 7)$ is the last solution.

⟨ **EDIT** ⟩ Here's such an elementary proof. For each e (other than the trivial $e = 2$), we choose some $f_0 > e$, try each f with $e < f_0 < f$, and then once $f \geq f_0$ we use the condition $F_n = 2^e + 2^f \equiv 2^e \bmod 2^f$ to get a congruence condition on n, and then reach a contradiction by considering F_n modulo some odd prime (usually 3, but with one much larger exception).

$e = 0$: We take $f_0 = 4$. Trying $f = 1$ and $f = 2$ yields the Fibonacci numbers $F_4 = 3$ and $F_5 = 5$, and $f = 3$ yields the non-Fibonacci number 9. Once $f \geq 4$ we have $F_n \equiv 1 \bmod 16$. But $F_n \bmod 16$ is periodic with period 24, and it turns out that the remainder is 1 only for $n \equiv 1, 2, 23 \bmod 24$. But $F_n \bmod 3$ has period 8, which is a factor of 24; and $F_1 = F_2 = F_{-1} = 1$. We deduce $F_n \equiv 1 \bmod 3$. Hence $2^f \equiv 0 \bmod 3$, which is impossible.

$e = 1$: The Fibonacci numbers F_n congruent to $2 \bmod 4$ are those with $n \equiv 3 \bmod 6$, and these always turn out to be $2 \bmod 32$. Thus $f \geq 5$, and $f = 5$ yields the Fibonacci number $34 = F_9$. We claim that this is the only possibility, using $f_0 = 6$. Once $f \geq 6$ we have $F_n \equiv 2 \bmod 64$, and then $n \equiv \pm 3 \bmod 24$. But (again thanks to 8-periodicity mod 3) this implies $F_n \equiv 2 \bmod 3$, so once more we reach a contradiction from the congruence $2^f \equiv 0 \bmod 3$.

$e = 2$: impossible because F_n is never $2 \bmod 4$.

$e = 3$: We take $f_0 = 5$. Since $2^3 + 2^4 = 24$ is not a Fibonacci number, we may assume $f \geq 5$, and then $F_n \equiv 8 \bmod 32$. This is equivalent to $n \equiv 6 \bmod 24$, which again yields a contradiction mod 3 since $2^f = F_n - 2^e$ would have to be a multiple of 3.

$e = 4$: This is the hardest case: because $f = 7$ yields $144 = F_{12}$, it is not enough to use congruences that can be deduced from $F_n \equiv 2 \bmod 2^7$, and we must take $f_0 > 7$. It turns out that $f_0 = 9$ works. Then $f = 5, 6, 8$ yield the non-Fibonacci $48, 80, 272$. Once $f \geq 9$ we must have $F_n \equiv 16 \bmod 2^9$. Now $F_n \bmod 2^9$ has period 768, but the condition $F_n \equiv 16 \bmod 2^9$ determines $n \bmod 384$ (half of 768), and we compute $n \equiv -84 \bmod 384$. Now $n \bmod 384$ determines F_n modulo the prime 4481 (the period is 128), and we find $F_n \equiv 2284 \bmod 4481$, whence $2^f = F_n - 2^e \equiv 2284 - 16 = 2268 \bmod 4481$. But this is impossible because 2 is a fourth power (even an 8th power) mod 4481, and 2268 is not.

⟨ **/EDIT** ⟩

But I doubt that one can prove that such a technique can work for all x...