On the number of aperiodic chiral bracelets of two colors

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Contents

1 The first formula for the g.f. of the number of aperiodic chiral bracelets of two colors

Let $a_k(n)$ be the number of bracelets (turnover necklaces) of length n that have no reflection symmetry and consist of k white beads and $n-k$ black beads. Such bracelets (with no reflection symmetry) are also called *chiral*.

Herbert Kociemba has proved that, for fixed $k \in \mathbb{Z}_{>0}$, the generating function of the sequence $(a_k(n) : n \in \mathbb{Z}_{>0})$ is given by

$$f_k(x) = \sum_{n=1}^{\infty} a_k(n)\, x^n = \frac{x^k}{2} \left(\frac{1}{k} \sum_{m|k} \frac{\phi(m)}{(1-x^m)^{k/m}} - \frac{1+x}{(1-x^2)^{\lfloor \frac{k}{2}+1 \rfloor}} \right), \tag{1}$$

where $\phi(\cdot)$ is Euler's totient function. See, for example, the documentation of the following sequences in the OEIS: A008804, A032246, A032247, A032248, A032249, and A032250.

Note that, unlike Bower [1] (in the documentation of the **DHK** transform), we trivially assume that all bracelets of length 1 or 2 do have reflection symmetry. Thus, we trivially have

$$a_k(1) = 0 = a_k(2) \quad \text{for all } k \in \mathbb{Z}_{>0},$$

and this is reflected in Kociemba's formula (1) above. This is because, for a bracelet with 1 bead, we may imagine an axis of symmetry passing through the single bead, while for a bracelet of length 2, we may imagine an axis of symmetry passing through the two beads (assuming they are placed diametrically opposite of each other on a circle). If a bracelet of length 2 has two beads of identical color, we may also consider an axis of symmetry going between these two beads (to the left and to the right of each one of them).

Let $b_k(n)$ be the number of *aperiodic* bracelets (turnover necklaces) of length n that have no reflection symmetry and consist of k white beads and $n-k$ black beads. Using the generating function $f_k(x)$ in Kocienba's formula (1) above, we prove that, for fixed $k \in \mathbb{Z}_{>0}$, the generating function of the sequence $(b_k(n) : n \in \mathbb{Z}_{>0})$ is given by

$$g_k(x) = \sum_{n=1}^{\infty} b_k(n)\, x^n = \sum_{d|k} \mu(d)\, f_{\frac{k}{d}}(x^d), \tag{2}$$

where $\mu(\cdot)$ is the Möbius function. In Section 3 of this note, we prove a more explicit formula for $g_k(x)$ (see equation (10)).

Equation (2) can be established if we prove either one of the following two equivalent formulas:

$$a_k(n) = \sum_{d|\gcd(n,k)} b_{\frac{k}{d}}\left(\frac{n}{d}\right) \quad \text{and} \quad b_k(n) = \sum_{d|\gcd(n,k)} \mu(d)\, a_{\frac{k}{d}}\left(\frac{n}{d}\right) \quad (k, n \in \mathbb{Z}_{>0}). \tag{3}$$

(Note that $a_k(n) = 0 = b_k(n)$ when $0 < n < k$.)

Proof of equation (2) from equations (3): Using equations (3), we get

$$g_k(x) = \sum_{n=1}^{\infty} b_k(n)\, x^n = \sum_{n=1}^{\infty} \left(\sum_{d|\,\gcd(n,k)} \mu(d)\, a_{\frac{k}{d}}\left(\frac{n}{d}\right) \right) x^n.$$

Letting $m = n/d$, we get

$$g_k(x) = \sum_{m=1}^{\infty} \left(\sum_{d|k} \mu(d)\, a_{\frac{k}{d}}(m) \right) x^{md} = \sum_{d|k} \mu(d) \left(\sum_{m=1}^{\infty} a_{\frac{k}{d}}(m) \left(x^d\right)^m \right) = \sum_{k|d} \mu(d)\, f_{\frac{k}{d}}(x^d),$$

which establishes equation (2). ■

For each $h \in \mathbb{Z}_{>0}$, let $a_k(n; h)$ be the number of bracelets (turnover necklaces) of length n and period h that have no reflection symmetry and consist of k white beads and $n-k$ black beads. Equations (3) can be established if we prove the following equalities:

$$a_k(n; d) = b_{\frac{k}{d}}\left(\frac{n}{d}\right) \quad \text{for all } n, k, d \in \mathbb{Z}_{>0} \text{ with } d|\gcd(n,k). \tag{4}$$

Proof of equations (3) from equations (4): It is sufficient to prove only the first one of equations (3) (since the first one implies the second one). Note also that the period d of a bracelet of length n that has no reflection symmetry and consists of k white beads and $n-k$ black beads should divide both n and k (and thus, $n-k$ as well). It thus follows from equations (4) that, for $n, k \in \mathbb{Z}_{>0}$,

$$a_k(n) = \sum_{d|\,\gcd(n,k)} a_k(n; d) = \sum_{d|\,\gcd(n,k)} b_{\frac{k}{d}}\left(\frac{n}{d}\right).$$

This establishes equations (3). ■

For $d = 1$, equations (4) are obvious. The most difficult part of this note is establishing equations (4) when $d \geq 2$. We essentially have to prove that, if a bracelet of length n/d with k/d white beads and $(n-k)/d$ black beads has a reflection symmetry, then a bracelet that consists of d copies of this bracelet also has a reflection symmetry. We also have to prove the converse: if a bracelet of length n and period d consists of k white beads and $n-k$ black beads and has a reflection property, then there is a contiguous part of it of length n/d that consists of k/d white beads and $(n-k)/d$ black beads, has a reflection property, and when repeated d times produces the original bracelet.

Proof of equations (4): Assume $d|\gcd(n,k)$ and $d \geq 2$. We consider two cases: (a) n/d is odd, and (b) n/d is even.

Case (a): n/d **is odd**. If $n/d = 1$, then a bracelet of length n and period $d = n$ consists of n white beads and 0 black beads (i.e., $k = d = n$). Such a bracelet obviously has a reflection property, and so does a bracelet of length $n/d = 1$ consisting of a bead of the same color as the beads in the original bracelet. The converse is also true.

If $n/d > 1$, consider a bracelet of length n/d with k/d white beads and $(n-k)/d$ black beads that has reflection symmetry; say its beads are $c_1, \dots, c_s, b, c_s, \dots, c_1$, where $s = \frac{(n/d)-1}{2}$. It obviously has an axis of symmetry through b and the middle of the two beads c_1. Now, suppose we repeat it d times to create a bracelet of length n, which obviously would have k white beads and $n-k$ black beads. Starting from one copy, we name the copies (going in one direction) $1, 2, \dots, d$.

If d is even, then the bracelet of length n has an axis of symmetry going through beads b of copies 1 and $\frac{d}{2}+1$. It also has another axis of symmetry going between the two consecutive c_1 beads of copies d and 1 and between the consecutive c_1 beads of copies $\frac{d}{2}$ and $\frac{d}{2}+1$. (If $d = 2$, then obviously $d = \frac{d}{2}+1$ and $1 = \frac{d}{2}$.) It thus has a reflection symmetry.

If d is odd ≥ 3, then the bracelet of length n has an axis of symmetry going through bead b of copy 1 and through the middle of two (consecutive) beads c_1 of copies $\frac{d+1}{2}$ and $\frac{d+3}{2}$ (and thus, it has a reflection symmetry).

We may also consider the converse: start with a bracelet of length n (with $n > d$ and n/d odd) that has a reflection symmetry, has period d, and consists of k white beads and $(n-k)/d$ beads. We may prove (using a similar argument as above) that it can be generated by a bracelet of length n/d that has a reflection symmetry and consists of k/d white beads and $(n-k)/d$ black beads. The proof of the converse is actually more complicated, but we omit the details. (A complication in the proof of the converse arises from the fact that a bracelet with a reflection property and an even number of beads may have more than one axes of symmetry.)

Case (b): n/d **is even**. Consider a bracelet of length n/d that has a reflection symmetry and consists of k/d white beads and $(n-k)/d$ black beads. Then its beats are either of the form $c_1, \dots, c_s, c_s, \dots, c_1$, where $s = \frac{n}{2d}$ (with an axis of symmetry going between the two c_s beads and between the two c_1 beads, or of the form $c_1, \dots, c_{s-1}, e_1, c_{s-1}, \dots, c_1, e_2$, where $s = \frac{n}{2d} - 1$ (with an axis of symmetry going through beads e_1 and e_2).

Now consider a bracelet that consists of d copies of the bracelet of length n/d described above. It obviously has length n and consists of k white beads and $n-k$ black beads. Starting from one copy, number the copies in one direction $1, 2, \dots, d$.

If the bracelet of length n/d is of the form $c_1, \dots, c_s, c_s, \dots, c_1$, then the bracelet of length n has an axis of symmetry going between the two consecutive c_s beads of copy 1 and the two consecutive c_s beads of copy $\frac{d}{2}+1$ if d is even, or going between the two consecutive c_s beads of copy 1 and the two consecutive c_1 beads of copies $\frac{d+1}{2}$ and $\frac{d+3}{2}$ if d is odd.

If the bracelet of length n/d is of the form $c_1, \ldots, c_{s-1}, e_1, c_{s-1}, \ldots, c_1, e_2$, then the bracelet of length n has an axis of symmetry going through e_1 of copy 1 and e_1 of copy $\frac{d}{2}+1$ (and another one going through e_2 of copy 1 and e_2 of copy $\frac{d}{2}+1$) if d is even; or has an axis of symmetry going through e_1 in copy 1 and through e_2 of copy $\frac{d+1}{2}$ if d is odd.

We may also consider the converse: start with a bracelet of length n (with n/d even) that has a reflection symmetry, has period d, and consists of k white beads and $(n-k)/d$ beads. We may prove (using a similar argument as above) that it can be generated by a bracelet of length n/d that has a reflection symmetry and consists of k/d white beads and $(n-k)/d$ black beads. Again, we omit the details.

Conclusion: Thus, given $n, k, d \in \mathbb{Z}_{>0}$ with $d|\gcd(n,k)$, we may establish a one-to-one correspondence between the collection of bracelets of length n and period d which have k white beads and $n-k$ black beads and a reflection symmetry and the collection of aperiodic bracelets of length n/d which have k/d white beads and $(n-k)/d$ black beads and a reflection symmetry. This proves that $a_k(n;d) = b_{\frac{k}{d}}\left(\frac{n}{d}\right)$. ■

2 Some formulas for the quantities $a_k(n)$ and $b_k(n)$

Recall that $a_k(n)$ is the number of chiral bracelets of length n that have k white beads and $n-k$ beads. Also, $b_k(n)$ is the number of aperiodic chiral bracelets of length n that have k white beads and $n-k$ beads. We know from equations (3) that

$$a_k(n) = \sum_{d|\gcd(n,k)} b_{\frac{k}{d}}\left(\frac{n}{d}\right) \quad \text{and} \quad b_k(n) = \sum_{d|\gcd(n,k)} \mu(d)\, a_{\frac{k}{d}}\left(\frac{n}{d}\right) \quad (k, n \in \mathbb{Z}_{>0}). \tag{5}$$

We also have $a_k(n) = 0 = b_k(n)$ when $0 < n < k$.

The following formulas are implicit in the comments of John P. McSorley for sequence A180472 and the paper by McSorley and Shoen [2]. For $1 \le k \le n$,

$$a_k(n) = -\frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor - (k \bmod 2)[1-(n \bmod 2)]}{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2n}\sum_{d|\gcd(n,k)} \phi(d)\binom{\frac{n}{d}}{\frac{k}{d}} \tag{6}$$

$$= -\frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor - (k \bmod 2)[1-(n \bmod 2)]}{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2k}\sum_{d|\gcd(n,k)} \phi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1}. \tag{7}$$

Obviously, equation (6) follows easily from equation (7). Given the results of this note, the easiest way to prove equation (7) is to use Herbert Kociemba's formula (1).

Proof of equation (7) from equation (1): Let

$$A_k(n) := \sum_{d|\gcd(n,k)} \phi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1} \quad \text{and} \quad B_k(n) := \binom{\lfloor\frac{n}{2}\rfloor - (k \bmod 2)[1-(n \bmod 2)]}{\lfloor\frac{k}{2}\rfloor}.$$

Equation (7) would follow easily from equation (1) if we establish that, for $k \geq 1$,

$$\sum_{n=1}^{\infty} A_k(n)\, x^n = x^k \sum_{d|k} \frac{\phi(d)}{(1-x^d)^{k/d}} \quad \text{and} \quad \sum_{n=1}^{\infty} B_k(n)\, x^n = \frac{(1+x)x^k}{(1-x^2)^{\lfloor\frac{k}{2}+1\rfloor}}. \tag{8}$$

It is well-known that, for any real $t > 0$ and $|x| < 1$, we have

$$(1-x)^{-t} = \sum_{s=0}^{\infty} \binom{-t}{s}(-x)^s = \sum_{s=0}^{\infty} \binom{t+s-1}{s} x^s. \tag{9}$$

We then have

$$\begin{aligned}
\sum_{n=1}^{\infty} A_k(n)\, x^n &= \sum_{n=1}^{\infty} \sum_{d|\gcd(n,k)} \phi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1} x^n \quad (\text{let } n = md) \\
&= \sum_{d|k} \phi(d) \sum_{m=1}^{\infty} \binom{m-1}{\frac{k}{d}-1} x^{md} \\
&= \sum_{d|k} \phi(d) \sum_{m=k/d}^{\infty} \binom{m-1}{\frac{k}{d}-1} (x^d)^m \quad (\text{let } s = m-(k/d)) \\
&= \sum_{d|k} \phi(d) \sum_{s=0}^{\infty} \binom{s+\frac{k}{d}-1}{s} (x^d)^{s+(k/d)} = x^k \sum_{d|k} \frac{\phi(d)}{(1-x^d)^{k/d}}.
\end{aligned}$$

This proves the first equation in (8).

To prove the second equation in (8), we consider two cases.

Case 1: k is even. Say $k = 2\nu$, where $\nu \in \mathbb{Z}_{>0}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} B_k(n)\, x^n &= \sum_{n=1}^{\infty} \binom{\lfloor\frac{n}{2}\rfloor}{\nu} x^n = \sum_{m=1}^{\infty} \binom{m}{\nu} x^{2m} + \sum_{m=0}^{\infty} \binom{m}{\nu} x^{2m+1} \\
&= \sum_{m=\nu}^{\infty} \binom{m}{\nu} x^{2m} + \sum_{m=\nu}^{\infty} \binom{m}{\nu} x^{2m+1} \quad (\text{let } s = m-\nu) \\
&= (1+x) \sum_{s=0}^{\infty} \binom{(\nu+1)+s-1}{s} (x^2)^{s+\nu} \\
&= \frac{x^{2\nu}(1+x)}{(1-x^2)^{\nu+1}} = \frac{(1+x)x^k}{(1-x^2)^{\lfloor\frac{k}{2}+1\rfloor}}.
\end{aligned}$$

Case 2: k is odd. Say $k = 2\nu + 1$, where $\nu \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} B_k(n)\, x^n &= \sum_{n=1}^{\infty} \binom{\lfloor \frac{n}{2} \rfloor - 1 + (n \bmod 2)}{\nu} x^n \\ &= \sum_{m=1}^{\infty} \binom{m-1}{\nu} x^{2m} + \sum_{m=0}^{\infty} \binom{m}{\nu} x^{2m+1} \\ &= \sum_{m=\nu+1}^{\infty} \binom{m-1}{\nu} x^{2m} + \sum_{m=\nu}^{\infty} \binom{m}{\nu} x^{2m+1}.\end{aligned}$$

We let $s = m - \nu - 1$ in the first sum and $s = m - \nu$ in the second one. We thus get

$$\begin{aligned}\sum_{n=1}^{\infty} B_k(n)\, x^n &= \sum_{s=0}^{\infty} \binom{s+\nu}{\nu} x^{2s+2\nu+2} + \sum_{s=0}^{\infty} \binom{s+\nu}{\nu} x^{2s+2\nu+1} \\ &= x^{2\nu+1}(x+1) \sum_{s=0}^{\infty} \binom{(\nu+1)+s-1}{s} (x^2)^s \\ &= \frac{x^{2\nu+1}(1+x)}{(1-x^2)^{\nu+1}} = \frac{(1+x)x^k}{(1-x^2)^{\lfloor \frac{k}{2}+1 \rfloor}}.\end{aligned}$$

This completes the proof of the second equation in (8). ■

Since we have two formulas for $a_k(n)$, given by equations (6) and (7), we may easily get a formula for $b_k(n)$, using the second equation in (5). But this would be a formula that is too complicated (and we do not even type it down in this note!). In Section 4 of this note, we prove a simpler formula for $b_k(n)$.

3 The second formula for the g.f. of the number of aperiodic chiral bracelets of two colors

Using equations (1) and (2) from Section 1, we may establish another formula for the generating function of the number of *aperiodic* bracelets of length n that have no reflection symmetry and consist of k white beads and $n - k$ black beads:

$$g_k(x) = \sum_{n=1}^{\infty} b_k(n)\, x^n = \frac{x^k}{2k} \sum_{d|k} \mu(d) \left(\frac{1}{(1-x^d)^{k/d}} - \frac{k(1+x^d)}{(1-x^{2d})^{\lfloor \frac{k}{2d}+1 \rfloor}} \right). \tag{10}$$

Proof of equation (10): Using equations (1) and (2), we get

$$\begin{aligned}
g_k(x) &= \sum_{d|k} \mu(d)\, f_{\frac{k}{d}}(x^d) \\
&= \sum_{d|k} \mu(d) \frac{(x^d)^{\frac{k}{d}}}{2} \left(\frac{d}{k} \sum_{m|(k/d)} \frac{\phi(m)}{(1-x^{dm})^{k/(dm)}} - \frac{1+x^d}{(1-x^{2d})^{\lfloor \frac{k}{2d}+1 \rfloor}} \right) \\
&= \frac{x^k}{2k} \sum_{d|k} d\mu(d) \sum_{m|(k/d)} \frac{\phi(m)}{(1-x^{dm})^{k/(dm)}} - \frac{x^k}{2k} \sum_{d|k} \mu(d)\, \frac{k(1+x^d))}{(1-x^{2d})^{\lfloor \frac{k}{2d}+1 \rfloor}}.
\end{aligned}$$

To finish the proof of equation (10), we need to show that

$$\sum_{d|k} d\mu(d) \sum_{m|(k/d)} \frac{\phi(m)}{(1-x^{dm})^{k/(dm)}} = \sum_{d|k} \frac{\mu(d)}{(1-x^d)^{k/d}}. \tag{11}$$

Using the associative property of Dirichlet convolutions, we get

$$\begin{aligned}
\sum_{d|k} d\mu(d) \sum_{m|(k/d)} \frac{\phi(m)}{(1-y^{dm/k})^{k/(dm)}} &= \sum_{d|k} d\mu(d) \sum_{m|(k/d)} \frac{\phi(m)}{(1-y^{m/(k/d)})^{(k/d)/m}} \\
&= \sum_{d|k} \left(\sum_{m|d} m\mu(m)\phi\left(\frac{d}{m}\right) \right) \frac{1}{(1-y^{d/k})^{k/d}}. \qquad (12)
\end{aligned}$$

We claim that

$$\sum_{m|d} m\mu(m)\phi\left(\frac{d}{m}\right) = \mu(d) \quad \text{for all } d \in \mathbb{Z}_{>0}. \tag{13}$$

Indeed, it is well-known that

$$\frac{\phi(d)}{d} = \sum_{m|d} \frac{\mu(m)}{m} \quad \text{for all } d \in \mathbb{Z}_{>0},$$

from which, by Möbius inversion, we get

$$\sum_{m|d} \mu(m) \frac{\phi(d/m)}{d/m} = \frac{\mu(d)}{d} \quad \text{for all } d \in \mathbb{Z}_{>0}.$$

The last equality is equivalent to equation (13).

From equations (12) and (13) above, we get

$$\sum_{d|k} d\mu(d) \sum_{m|(k/d)} \frac{\phi(m)}{(1-y^{dm/k})^{k/(dm)}} = \sum_{d|k} \frac{\mu(d)}{(1-y^{d/k})^{k/d}}.$$

Letting $y = x^k$ in the above equation, we get equation (11), and this finishes the proof of equation (10). ■

4 A simpler formula for $b_k(n)$

At the end of Section 2 of this note, we found a formula for $b_k(n)$, which is the number of aperiodic chiral bracelets of length n that have k white beads and $n-k$ black beads. The formula, however, was too complicated to even type it down. Thus, here, we give a simpler one.

For $n, k, d \in \mathbb{Z}_{>0}$ with $d \mid \gcd(n,k)$, let

$$c(n,k,d) := \frac{n}{d} + \frac{(-1)^{\frac{k}{d}} - 1}{2}.$$

Then

$$b_k(n) = \frac{1}{2k} \sum_{d \mid \gcd(n,k)} \mu(d) \left(\binom{\frac{n}{d}-1}{\frac{k}{d}-1} - k \binom{\left\lfloor \frac{c(n,k,d)}{2} \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor} \right). \tag{14}$$

Since $\binom{\alpha}{\beta} = 0$ for $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ with $\alpha < \beta$, the above formula is true even when $0 < n < k$, in which case, we trivially have $b_k(n) = 0$.

Proof of equation (14) from equation (10): Let

$$C_k(n) := \sum_{d \mid \gcd(n,k)} \mu(d) \binom{\frac{n}{d}-1}{\frac{k}{d}-1} \quad \text{and} \quad D_k(n) := \sum_{d \mid \gcd(n,k)} \mu(d) \binom{\left\lfloor \frac{c(n,k,d)}{2} \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor}.$$

Equation (14) would follow easily from equation (10) if we establish that, for $k \geq 1$,

$$\sum_{n=1}^{\infty} C_k(n)\, x^n = x^k \sum_{d \mid k} \frac{\mu(d)}{(1-x^d)^{k/d}} \quad \text{and} \quad \sum_{n=1}^{\infty} D_k(n)\, x^n = x^k \sum_{d \mid k} \frac{\mu(d)(1+x^d)}{(1-x^{2d})^{\left\lfloor \frac{k}{2d}+1 \right\rfloor}}. \tag{15}$$

Because of equations (9), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} C_k(n)\, x^n &= \sum_{n=1}^{\infty} \sum_{d \mid \gcd(n,k)} \mu(d) \binom{\frac{n}{d}-1}{\frac{k}{d}-1} x^n \quad (\text{let } n = md) \\
&= \sum_{d \mid k} \mu(d) \sum_{m=1}^{\infty} \binom{m-1}{\frac{k}{d}-1} x^{md} \\
&= \sum_{d \mid k} \mu(d) \sum_{m=k/d}^{\infty} \binom{m-1}{\frac{k}{d}-1} (x^d)^m \quad (\text{let } s = m - (k/d)) \\
&= \sum_{d \mid k} \mu(d) \sum_{s=0}^{\infty} \binom{s+\frac{k}{d}-1}{s} (x^d)^{s+(k/d)} = x^k \sum_{d \mid k} \frac{\mu(d)}{(1-x^d)^{k/d}}.
\end{aligned}$$

This proves the first equation in (15).

To prove the second equation in (15), notice first that

$$\sum_{n=1}^{\infty} D_k(n)\, x^n = \sum_{n=1}^{\infty} \sum_{d|\,\gcd(n,k)} \mu(d) \binom{\left\lfloor \frac{c(n,k,d)}{2} \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor} x^n = \sum_{d|k} \mu(d) E(k,d,x),$$

where

$$E(k,d,x) := \sum_{m=1}^{\infty} \binom{\left\lfloor \frac{m+\frac{(-1)^{k/d}-1}{2}}{2} \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor} x^{md}$$

for all $k, d \in \mathbb{Z}_{>0}$ with $d|k$. If we prove that

$$E(k,d,x) = \frac{x^k(1+x^d)}{(1-x^{2d})^{\left\lfloor \frac{k}{2d}+1 \right\rfloor}} \quad \text{for } d|k, \tag{16}$$

then the second equation in (15) would follow immediately.

We consider two cases. In each case, we use equations (9).

Case 1: k/d is even. Then

$$\begin{aligned}
E(k,d,x) &= \sum_{m=1}^{\infty} \binom{\left\lfloor \frac{m}{2} \right\rfloor}{\frac{k}{2d}} x^{md} = \sum_{s=1}^{\infty} \binom{s}{\frac{k}{2d}} x^{2sd} + \sum_{s=0}^{\infty} \binom{s}{\frac{k}{2d}} x^{2sd+d} \\
&= \sum_{s=k/(2d)}^{\infty} \binom{s}{\frac{k}{2d}} x^{2sd} + \sum_{s=k/(2d)}^{\infty} \binom{s}{\frac{k}{2d}} x^{2sd+d} \quad (\text{let } r = s - \tfrac{k}{2d}) \\
&= (1+x^d) \sum_{r=0}^{\infty} \binom{r+\frac{k}{2d}}{\frac{k}{2d}} \left(x^{2d}\right)^{r+\frac{k}{2d}} \\
&= x^k(1+x^d) \sum_{r=0}^{\infty} \binom{\left(\frac{k}{2d}+1\right)+r-1}{r} \left(x^{2d}\right)^r \\
&= \frac{x^k(1+x^d)}{(1-x^{2d})^{\frac{k}{2d}+1}} = \frac{x^k(1+x^d)}{(1-x^{2d})^{\left\lfloor \frac{k}{2d}+1 \right\rfloor}}.
\end{aligned}$$

Case 2: k/d is odd. Then

$$\begin{aligned}
E(k,d,x) &= \sum_{m=1}^{\infty} \binom{\left\lfloor \frac{m-1}{2} \right\rfloor}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{md} \\
&= \sum_{s=1}^{\infty} \binom{s-1}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{2sd} + \sum_{s=0}^{\infty} \binom{s}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{2sd+d} \\
&= \sum_{s=\frac{1}{2}\left(\frac{k}{d}+1\right)}^{\infty} \binom{s-1}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{2sd} + \sum_{s=\frac{1}{2}\left(\frac{k}{d}-1\right)}^{\infty} \binom{s}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{2sd+d}.
\end{aligned}$$

We let $r = s - \frac{1}{2}\left(\frac{k}{d}+1\right)$ in the first sum and $r = s - \frac{1}{2}\left(\frac{k}{d}-1\right)$ in the second one. We thus get

$$\begin{aligned} E(k,d,x) &= \sum_{r=0}^{\infty} \binom{r+\frac{1}{2}\left(\frac{k}{d}-1\right)}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{2rd+k+d} + \sum_{r=0}^{\infty} \binom{r+\frac{1}{2}\left(\frac{k}{d}-1\right)}{\frac{1}{2}\left(\frac{k}{d}-1\right)} x^{2rd+k}. \\ &= x^k(1+x^d) \sum_{r=0}^{\infty} \binom{\frac{1}{2}\left(\frac{k}{d}+1\right)+r-1}{r} (x^{2d})^r \\ &= \frac{x^k(1+x^d)}{(1-x^{2d})^{\frac{1}{2}\left(\frac{k}{d}+1\right)}} = \frac{x^k(1+x^d)}{(1-x^{2d})^{\left\lfloor \frac{k}{2d}+1 \right\rfloor}}. \end{aligned}$$

Putting both cases together allows us to prove equation (16), from which the second equation in (15) follows immediately. ■

5 Formulas for the total number of chiral bracelets of two colors

Let $a(n)$ be the total number of chiral bracelets (bracelets with no reflection symmetry) of two colors, and let $b(n)$ be the total number of aperiodic chiral braceletes of two colors. In other words,

$$a(n) = \sum_{k=1}^{n} a_k(n) \quad \text{and} \quad b(n) = \sum_{k=1}^{n} b_k(n).$$

Let also $f(x)$ and $g(x)$ be their corresponding generating functions; that is,

$$f(x) = \sum_{n=1}^{\infty} a(n)\, x^n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} b(n)\, x^n.$$

Using the results of the previous sections, we shall prove in this section that

$$a(n) = -2^{\left\lfloor \frac{n}{2}-3 \right\rfloor}(7-(-1)^n) + \frac{1}{2n}\sum_{d|n} \phi(d) 2^{\frac{n}{d}}; \tag{17}$$

$$b(n) = \frac{1}{2}\sum_{d|n} \mu(d) \left(\frac{2^{\frac{n}{d}}}{n} - 2^{\left\lfloor \frac{n}{2d}-2 \right\rfloor} \left(7-(-1)^{\frac{n}{d}}\right) \right); \tag{18}$$

$$f(x) = \frac{1}{2}\left(-\frac{x(2+3x)}{1-2x^2} - \sum_{m=1}^{\infty} \frac{\phi(m)}{m} \log(1-2x^m) \right); \tag{19}$$

$$g(x) = \frac{1}{2}\sum_{m=1}^{\infty} \mu(m) \left(-\frac{x^m(2+3x^m)}{1-2x^{2m}} - \frac{1}{m}\log(1-2x^m) \right). \tag{20}$$

Formulas (17) and (18) above are special cases of more general formulas due to Robert A. Russell that appear in the documentation of sequences A059076 and A032239, respectively. In addition, formulas (19) and (20) above are special cases of more general formulas due to Herbert Kociemba that appear in the documentation of sequences A059076 and A032239, respectively.

Proof of equation (17): From equation (6), we get

$$a(n) = \sum_{k=1}^{n} a_k(n) = \sum_{k=1}^{n} (-\alpha(n,k) + \beta(n,k)),$$

where

$$\alpha(n,k) := \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor - (k \bmod 2)[1-(n \bmod 2)]}{\lfloor \frac{k}{2} \rfloor} \quad \text{and} \quad \beta(n,k) := \frac{1}{2n} \sum_{d \mid \gcd(n,k)} \phi(d) \binom{\frac{n}{d}}{\frac{k}{d}}.$$

To finish the proof of equation (17), we need to show that

$$\sum_{k=1}^{n} \alpha(n,k) = 2^{\lfloor \frac{n}{2} - 3 \rfloor}(7-(-1)^n) - \frac{1}{2} \quad \text{and} \quad \sum_{k=1}^{n} \beta(n,k) = \frac{1}{2n} \sum_{d \mid n} \phi(d) 2^{n/d} - \frac{1}{2}. \tag{21}$$

To prove the first equation in (21), we consider two cases. When n is even, say $n = 2v$, where $v \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \alpha(n,k) &= \frac{1}{2} \sum_{k=1}^{2v} \binom{v - (k \bmod 2)}{\lfloor \frac{k}{2} \rfloor} \\
&= \frac{1}{2} \left(\sum_{\ell=1}^{v} \binom{v}{\ell} + \sum_{\ell=0}^{v-1} \binom{v-1}{\ell} \right) \\
&= 3 \cdot 2^{v-2} - \frac{1}{2} = 2^{\lfloor \frac{n}{2} - 3 \rfloor}(7-(-1)^n) - \frac{1}{2}.
\end{aligned}$$

When n is odd, say $n = 2v+1$, where $v \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \alpha(n,k) &= \frac{1}{2} \sum_{k=1}^{2v+1} \binom{v}{\lfloor \frac{k}{2} \rfloor} \\
&= \frac{1}{2} \left(\sum_{\ell=1}^{v} \binom{v}{\ell} + \sum_{\ell=0}^{v} \binom{v}{\ell} \right) \\
&= 2^{v} - \frac{1}{2} = 2^{\lfloor \frac{n}{2} - 3 \rfloor}(7-(-1)^n) - \frac{1}{2}.
\end{aligned}$$

Finally, we prove the second equation in (21):

$$\begin{aligned}
\sum_{k=1}^{n} \beta(n,k) &= \sum_{k=1}^{n} \frac{1}{2n} \sum_{d \mid \gcd(n,k)} \phi(d) \binom{\frac{n}{d}}{\frac{k}{d}} \\
&= \frac{1}{2n} \sum_{d \mid n} \phi(d) \sum_{m=1}^{n/d} \binom{\frac{n}{d}}{m} \\
&= \frac{1}{2n} \sum_{d \mid n} \phi(d)(2^{n/d} - 1) = \frac{1}{2n} \sum_{d \mid n} \phi(d) 2^{n/d} - \frac{1}{2}.
\end{aligned}$$

This completes the proof of equation (17). ■

Proof of equation (18): From equation (14), we get

$$b(n) = \sum_{k=1}^{n} b_k(n) = \sum_{k=1}^{n} (\gamma(n,k) - \delta(n,k)), \tag{22}$$

where

$$\gamma(n,k) := \frac{1}{2k} \sum_{d|\gcd(n,k)} \mu(d) \binom{\frac{n}{d}-1}{\frac{k}{d}-1} = \frac{1}{2n} \sum_{d|\gcd(n,k)} \mu(d) \binom{\frac{n}{d}}{\frac{k}{d}}$$

and

$$\delta(n,k) := \frac{1}{2} \sum_{d|\gcd(n,k)} \mu(d) \binom{\left\lfloor \frac{c(n,k,d)}{2} \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor},$$

where

$$c(n,k,d) := \frac{n}{d} + \frac{(-1)^{\frac{k}{d}} - 1}{2} \quad \text{for } d|\gcd(n,k).$$

To prove equation (18), we first need to establish that

$$\sum_{k=1}^{n} \gamma(n,k) = \frac{1}{2n} \sum_{d|n} \mu(d)(2^{n/d} - 1) \tag{23}$$

and

$$\sum_{k=1}^{n} \delta(n,k) = \frac{1}{2} \sum_{d|n} \mu(d) \left(-1 + 2^{\left\lfloor \frac{n}{2d} - 2 \right\rfloor} (7 - (-1)^{n/d})\right). \tag{24}$$

We have

$$\sum_{k=1}^{n} \gamma(n,k) = \frac{1}{2n} \sum_{d|n} \mu(d) \sum_{m=1}^{n/d} \binom{\frac{n}{d}}{m} = \frac{1}{2n} \sum_{d|n} \mu(d)(2^{n/d} - 1),$$

which proves equation (23).

To prove equation (24), we first notice that

$$\begin{aligned} \sum_{k=1}^{n} \delta(n,k) &= \frac{1}{2} \sum_{k=1}^{n} \mu(d) \sum_{d|\gcd(n,k)} \binom{\left\lfloor \frac{c(n,k,d)}{2} \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor} \\ &= \frac{1}{2} \sum_{k=1}^{n} \mu(d) \sum_{d|\gcd(n,k)} \binom{\left\lfloor \frac{1}{2} \left(\frac{n}{d} + \frac{(-1)^{k/d} - 1}{2} \right) \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor} \\ &= \frac{1}{2} \sum_{d|n} \mu(d) \sum_{m=1}^{n/d} \binom{\left\lfloor \frac{1}{2} \left(\frac{n}{d} + \frac{(-1)^{m} - 1}{2} \right) \right\rfloor}{\left\lfloor \frac{m}{2} \right\rfloor}. \end{aligned} \tag{25}$$

Because of equation (25), to complete the proof of equation (24), we need to show that

$$\sum_{m=1}^{n/d} \binom{\left\lfloor \frac{1}{2}\left(\frac{n}{d}+\frac{(-1)^m-1}{2}\right)\right\rfloor}{\left\lfloor \frac{m}{2}\right\rfloor} = -1 + 2^{\left\lfloor \frac{n}{2d}-2\right\rfloor}(7-(-1)^{n/d}) \quad \text{for all } n,d \in \mathbb{Z}_{>0} \text{ with } d|n.$$

We consider two cases depending on the parity of n/d.

Case 1: n/d **is even.** In this case,

$$\begin{aligned}\sum_{m=1}^{n/d} \binom{\left\lfloor \frac{1}{2}\left(\frac{n}{d}+\frac{(-1)^m-1}{2}\right)\right\rfloor}{\left\lfloor \frac{m}{2}\right\rfloor} &= \sum_{\ell=1}^{\frac{n}{2d}} \binom{\frac{n}{2d}}{\ell} + \sum_{\ell=0}^{\frac{n}{2d}-1} \binom{\frac{n}{2d}-1}{\ell} \\ &= 3\cdot 2^{\frac{n}{2d}-1} - 1 = -1 + 2^{\left\lfloor \frac{n}{2d}-2\right\rfloor}(7-(-1)^{n/d}).\end{aligned}$$

Case 2: n/d **is odd.** In this case,

$$\begin{aligned}\sum_{m=1}^{n/d} \binom{\left\lfloor \frac{1}{2}\left(\frac{n}{d}+\frac{(-1)^m-1}{2}\right)\right\rfloor}{\left\lfloor \frac{m}{2}\right\rfloor} &= \sum_{\ell=1}^{\frac{n}{2d}-\frac{1}{2}} \binom{\frac{n}{2d}-\frac{1}{2}}{\ell} + \sum_{\ell=0}^{\frac{n}{2d}-\frac{1}{2}} \binom{\frac{n}{2d}-\frac{1}{2}}{\ell} \\ &= 2^{\frac{n}{2d}+\frac{1}{2}} - 1 = -1 + 2^{\left\lfloor \frac{n}{2d}-2\right\rfloor}(7-(-1)^{n/d}).\end{aligned}$$

This completes the proof of equation (24).

If $n > 1$, it is well-known that $\sum_{d|n} \mu(d) = 0$, in which case, equations (22), (23), and (24) immediately yield (18).

If $n = 1$, then

$$\gamma(1,1) = \frac{1}{2} = \delta(1,1) \Rightarrow b(1) = 0,$$

and we see that equation (18) is trivially satisfied. This completes the proof of equation (18). ∎

Proof of equation (19): It suffices to prove the following two identities:

$$\sum_{n=1}^{\infty} 2^{\left\lfloor \frac{n}{2}-3\right\rfloor}(7-(-1)^n)x^n = \frac{x(2+3x)}{2(1-2x^2)} \quad \text{and} \tag{26}$$

$$\sum_{n=1}^{\infty} \frac{1}{2n}\sum_{d|n} \phi(d) 2^{n/d} x^n = -\frac{1}{2}\sum_{d=1}^{\infty} \frac{\phi(d)}{d}\log(1-2x^d). \tag{27}$$

We have

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{\left\lfloor \frac{n}{2}-3\right\rfloor}(7-(-1)^n)x^n &= \sum_{m=1}^{\infty} 2^{m-3}(7-1)x^{2m} + \sum_{m=0}^{\infty} 2^{m-3}(7+1)x^{2m+1} \\ &= \frac{3x^2}{2(1-2x^2)} + \frac{x}{1-2x^2} = \frac{x(2+3x)}{2(1-2x^2)}.\end{aligned}$$

This proves equation (26). We also have

$$\sum_{n=1}^{\infty}\frac{1}{2n}\sum_{d|n}\phi(d)2^{n/d}x^n=\frac{1}{2}\sum_{d=1}^{\infty}\frac{\phi(d)}{d}\sum_{m=1}^{\infty}\frac{(2x^d)^m}{m}=-\frac{1}{2}\sum_{r=1}^{\infty}\frac{\phi(d)}{d}\log(1-2x^d).$$

The proves equation (27), and the proof of equation (19) is complete. ■

Proof of equation (20): It suffices to prove the following two identities:

$$\sum_{n=1}^{\infty}\sum_{d|n}\mu(d)2^{\lfloor\frac{n}{2d}-2\rfloor}\left(7-(-1)^{\frac{n}{d}}\right)x^n=\sum_{m=1}^{\infty}\mu(m)\frac{x^m(2+3x^m)}{1-2x^{2m}}\quad\text{and}\tag{28}$$

$$\sum_{n=1}^{\infty}\frac{1}{2n}\sum_{d|n}\mu(d)2^{n/d}x^n=-\frac{1}{2}\sum_{r=1}^{\infty}\frac{\mu(d)}{d}\log(1-2x^d).\tag{29}$$

We have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{d|n}\mu(d)2^{\lfloor\frac{n}{2d}-2\rfloor}\left(7-(-1)^{\frac{n}{d}}\right)x^n&=\sum_{d=1}^{\infty}\mu(d)\sum_{m=1}^{\infty}2^{\lfloor\frac{m}{2}-2\rfloor}\left(7-(-1)^m\right)(x^d)^m\\&=\sum_{d=1}^{\infty}\mu(d)\frac{x^d(2+3x^d)}{1-2x^{2d}}\quad\text{(from equation (26))}.\end{aligned}$$

This proves equation (28). The proof of equation (29) is similar to the proof of equation (27) above, and thus it is omitted. This completes the proof of equation (20). ■

6 Final remarks

It can be easily proved that, for $k\in\mathbb{Z}_{>0}$,

$$[a_k(n)=b_k(n)\quad\text{for all } n\in\mathbb{Z}_{>0}]\iff[k\in\{1,4\}\text{ or } k \text{ is a positive prime}].$$

We have

$$f_k(x)=g_k(x)=0\quad\text{for } k\in\{1,2\};\quad f_4(x)=g_4(x)=\frac{x^7}{(1-x)^4(1+x)^2(1+x^2)};$$

and

$$f_k(x)=g_k(x)=\frac{x^k}{2}\left(\frac{1}{k(1-x)^k}+\frac{k-1}{k(1-x^k)}-\frac{(1+x)}{(1-x^2)^{\frac{k+1}{2}}}\right)\quad\text{for } k \text{ odd prime}\geq 3.$$

Finally, we remark (one more time) that Bower [1] defined the **DHK** (bracelet, identity, unlabeled) transform of a sequence $(c(n):n\geq 1)$ as the sum of the $\mathbf{DHK}_k$ transforms of the sequence $(c(n) : n \geq 1)$ for k from 1 to n. (In his notation, k stands for the number of boxes that contain a total of n balls.)

Because he gave *special definitions* to the $\mathbf{DHK}_1$ and $\mathbf{DHK}_2$ transforms (different from the definition of the $\mathbf{DHK}_k$ transform of sequence $(c(n) : n \geq 1)$ for $k > 2$), we see that the values of $b(1)$ and $b(2)$ (which are both zero!) do not agree with A032239(1) and A032239(2) (which are 2 and 1, respectively).

References

[1] C. G. Bower, Further transformations of integer sequences, web article in *The on-line encyclopedia of integer sequences*, `https://oeis.org/transforms2.html`.

[2] J. P. McSorley and A. H. Schoen, Rhombic tilings of (n, k)-Ovals, (n, k, λ)-cyclic difference sets, and related topics, *Discrete Math.* **313** (2013), 129–154.