Proof of “empirical” recurrence for A189600

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Let $P(n)$ be the set of permutations counted by $a(n)$. For $n-8 \le k \le n-2$ or $k = n$. we define a one-to-one map $\Phi_{k,n}$ from $P(k)$ into $P(n)$ as follows: for $\pi \in P(k)$,

$$\begin{aligned} \Phi_{k,n}(\pi)_j &= \pi_j \text{ for } j \le k \\ &= n \text{ for } j = k+1 \\ &= j-1 \text{ otherwise} \end{aligned}$$

It is straightforward to verify that $\Phi_{k,n}(\pi) \in P(n)$: its displacements are those of π for $j \le k$, $k+1-n$ for n, and 1 for $k+1 \le j < n$. The ranges $\Phi_{k,n}(P(k))$ for different k are disjoint since $\pi_{k,n}(k+1) = n$. To complete the proof of $a(n) = a(n-1) + a(n-3) + \ldots + a(n-8)$ for $n > 8$, it suffices to show that every member of $P(n)$ is in $\Phi_{k,n}(P(k))$ for some k, $n-8 \le k \le n-2$ or $k = n$.

Consider a permutation $\pi \in P(n)$. The constraint on displacement says each $\pi(j)$ must be one of $j-1, j, j+2. \ldots, j+7$. In particular, n can occur in any of positions $n-7$ to $n-2$ or n. If it occurs in position i, then I claim $\pi \in \Phi_{i-1,n}(P(i-1))$. First of all, $\pi(n)$ must be $n-1$ or n, but if $i \ne n$, the value n is not available so only $\pi(n) = n-1$ is possible. Similarly, $\pi(n-1) \ge n-2$, but if $i < n-1$, the values $n-1$ and n are not available so $\pi(n-1) = n-2$. Continuing in this way, we get $\pi(j) = j-1$ for $i+1 \le j \le n$. This leaves π to map $\{1, \ldots, i-1\}$ to itself, and since the constraint on displacements is the same, the restriction of π to $\{1, \ldots, i-1\}$ is in $P(i-1)$, and π is the image of this permutation under $\Phi_{i-1,n}$. This completes the proof.