A Generating Function for Dimensions of Spaces of Siegel Cusp Forms of Genus 2 and Weight k

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Eie[1] gives a formula for the dimension of the space $S_k(\Gamma_2)$ of Siegel cusp forms of genus 2 and weight $k \geq 4$:

$$\dim S_k(\Gamma_2) = N_1 + N_2 + N_3 + N_4$$

where

$$N_1 = \begin{cases} 2^{-7}3^{-3} \times [1131, 229, -229, -1131, 427, -571, 123, -203, 203, -123, 571, -427] \\ \qquad\qquad \text{if} \qquad k \equiv [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] \mod 12 \end{cases}$$

$$N_2 = \begin{cases} 5^{-1} & k \equiv 0 \mod 5 \\ -5^{-1} & k \equiv 3 \mod 5 \\ 0 & \text{otherwise} \end{cases}$$

$$N_3 = \begin{cases} 2^{-5}3^{-3} \times [17k - 294, -25k + 325, -25k + 254, 17k - 261, 17k - 86, \\ \qquad -k + 53, -k - 42, -7k + 91, -7k + 2, -k - 27, -k + 166, 17k - 181] \\ \qquad \text{if} \qquad k \equiv [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] \mod 12 \end{cases}$$

and

$$N_4 = \begin{cases} 2^{-7}3^{-3}5^{-1}(2k^3 + 96k^2 - 52k - 3231) & \text{if } k \text{ is even} \\ 2^{-7}3^{-3}5^{-1}(2k^3 - 114k^2 + 2018k - 9051) & \text{if } k \text{ is odd} \end{cases}$$

Note that the above formula is only guaranteed correct for $k \geq 4$ (with the dimensions corresponding to $k \leq 3$ being zero), but this is easy to remedy once we obtain a generating function for all N_i.

To obtain a generating function for N_1, recall that the generating function for the constant sequence $\{1, 1, 1, \ldots\}$ is given by $\frac{1}{1-x}$. Similarly, a generating function for $\{1, 0, 1, 0, \ldots\}$ is given by $\frac{1}{1-x^2}$. It follows that a generating function f_1 for N_1 is given by

$$f_1(x) = 2^{-7}3^{-3} \left(1131 \frac{1}{1-x^{12}} + 229 \frac{x^1}{1-x^{12}} - 229 \frac{x^2}{1-x^{12}} + \ldots - 427 \frac{x^{11}}{1-x^{12}} \right)$$

$$= 2^{-7}3^{-3} \left(\frac{1131 + 1360x + 1131x^2 + 427x^4 - 144x^5 - 21x^6 - 224x^7 - 21x^8 - 144x^9 + 427x^{10}}{1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11}} \right)$$

A generating function f_2 for N_2 is much easier:

$$f_2(x) = 5^{-1} \left(\frac{1}{1-x^5} - \frac{x^3}{1-x^5} \right)$$

$$= 5^{-1} \left(\frac{1 + x + x^2}{1 + x + x^2 + x^3 + x^4} \right)$$

A generating function for N_3 requires the following observation: the sequence $\{1, 2, 3, 4, 5, \ldots\}$ is generated by $\frac{1}{(1-x)^2}$, any sequence $\{a(k)\}$ satisfying $a(k) = mk + b$ for constants m and b has a generating function which is a ratio of polynomials of order at most 2. It follows that N_3 has a generating function f_3 given by

$$f_3(x) = 2^{-5}3^{-3}\left(\frac{498x^{12} - 294}{(1-x^{12})^2} + \frac{-600x^{13} + 300x}{(1-x^{12})^2} + \ldots + \frac{198x^{23} + 6x^{11}}{(1-x^{12})^2}\right)$$

$$= 2^{-5}3^{-3}\left(\frac{-294 + 300x - 972x^2 + 990x^3 - 1554x^4 + 1608x^5 - 1428x^6 + 1554x^7 - 702x^8 + 816x^9 - 168x^{10} + 198x^{11}}{1 + 4x^2 + 8x^4 + 10x^6 + 8x^8 + 4x^{10} + x^{12}}\right)$$

Just as in the linear case, any sequence whose terms are given by a polynomial of degree m admits a generating function which is a ratio of poylnomials of degree at most $m+1$. The generating function of

$$2k^3 + 96k^2 - 52k - 3231$$

is

$$\frac{-3231 + 9739x - 9581x^2 + 3085x^3}{1 - 4x + 6x^2 - 4x^3 + x^4}$$

and the generating function of

$$2k^3 - 114k^2 + 2018k - 9051$$

is

$$\frac{-9051 + 29059x - 31181x^2 + 11185x^3}{1 - 4x + 6x^2 - 4x^3 + x^4}$$

so the generating function f_4 for N_4 is

$$f_4(x) = 2^{-7}3^{-3}5^{-1}\left(\frac{-3231 - 3914x + 13903x^2 + 10708x^3 - 20129x^4 - 8426x^5 + 11185x^6}{1 - x - 3x^2 + 3x^3 + 3x^4 - 3x^5 - x^6 + x^7}\right)$$

Compiling the above information, we have a generating function for $N_1 + N_2 + N_3 + N_4$:

$$f_1 + f_2 + f_3 + f_4 = \frac{-x^3 + x^7 + x^8 + x^9 + x^{10} - x^{13} - x^{14} - x^{17} + x^{18} - x^{19} - x^{21} + x^{24} + x^{26}}{1 - x^4 - x^5 - x^6 + x^9 + x^{10} + x^{11} - x^{12} - x^{15} + x^{16} + x^{17} + x^{18} - x^{21} - x^{22} - x^{23} + x^{27}}$$

$$= -x^3 + x^{10} + x^{12} + x^{14} + 2x^{16} + 2x^{18} + \ldots$$

Now we need only correct for the $k = 3$ case (dimensions ought not to be negative!) to obtain a true generating function for $\dim(S_k(\Gamma_2))$. That is,

$$\sum_k \dim(S_k(\Gamma_2))x^k = f_1 + f_2 + f_3 + f_4 + x^3$$

$$= \frac{x^{10} + x^{12} - x^{15} - x^{17} + x^{20} - x^{25} + x^{30}}{1 - x^4 - x^5 - x^6 + x^9 + x^{10} + x^{11} - x^{12} - x^{15} + x^{16} + x^{17} + x^{18} - x^{21} - x^{22} - x^{23} + x^{27}}$$

$$= \frac{x^{10}(1 + x^2 - x^5 - x^7 + x^{10} - x^{15} + x^{20})}{(-1+x)^4(1+x)^3(1 + 2x^2 + 2x^4 + x^6)^2(1 + x + x^4 + x^7 + x^8)}$$

References

[1] M. Eie, "Contributions from conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n, \mathbb{Z})$ to the dimension formula", Trans. Amer. Math. Soc. 285 (1984), no. 1, 408–409. Retrieved via https://www.jstor.org/stable/1999488.