Notes on A122193

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Name: Triangle $T(n,k)$ of number of loopless multigraphs with n labeled edges and k labeled vertices and without isolated vertices,$n \geq 1; 2 \leq k \leq 2n$.

We will compare the properties of A122193 with those of A131689, the triangle of numbers $t(n,k) := k!S(n,k)$, where $S(n,k)$ denotes the Stirling numbers of the second kind. The first few rows of these two arrays are shown below. We denote the row polynomials of A131689 by $r_n(x)$. These polynomials are variously known as Fubini polynomials, geometric polynomials or ordered Bell polynomials in the literature [Bo'05], [Bo'16], [DiKu'11]. We denote the row polynomials of A122193 by $R_n(x)$.

A131689 $t(n,k)$

$n\diagdown k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	2			
3	0	1	6	6		
4	0	1	14	36	24	
5	0	1	30	150	240	120
...						

A122193 $T(n,k)$

n$\diagdown k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	0	1						
2	0	0	1	6	6				
3	0	0	1	24	114	180	90		
4	0	0	1	78	978	4320	8460	7560	2520
...									

1) Double exponential generating functions.

A131689

$$\exp(-x)\sum_{n=0}^{\infty}\exp\left(\binom{n}{1}y\right)\frac{x^n}{n!} \quad = \quad \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(t(n,k)\frac{x^k}{k!}\right)\frac{y^n}{n!} \qquad (1)$$

A122193

$$\exp(-x)\sum_{n=0}^{\infty}\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!} \quad = \quad \sum_{n=0}^{\infty}\sum_{k=0}^{2n}\left(T(n,k)\frac{x^k}{k!}\right)\frac{y^n}{n!} \qquad (2)$$

The double e.g.f. for A131689 is equivalent to the e.g.f. $\exp(x(\exp(y)-1))$ for the Stirling numbers of the second kind. The expansion of the double e.g.f. for A122193 begins

$$\exp(-x)\sum_{n=0}^{\infty}\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!} = 1+\left(\frac{x^2}{2!}\right)\frac{y}{1!}+\left(\frac{x^2}{2!}+\frac{6x^3}{3!}+\frac{6x^4}{4!}\right)\frac{y^2}{2!} + \left(\frac{x^2}{2!}+\frac{24x^3}{3!}+\frac{114x^4}{4!}+\frac{180x^5}{5!}+\frac{90x^6}{6!}\right)\frac{y^3}{3!}+\cdots .$$

2) Recurrence equations for table entries.

A131689

$$t(n,k) = \binom{k}{1}\left(t(n-1,k)+t(n-1,k-1)\right) \tag{3}$$

A122193

$$T(n,k) = \binom{k}{2}\left(T(n-1,k)+2T(n-1,k-1)+T(n-1,k-2)\right) \tag{4}$$

Proof of the recurrence for A122193.

Let $A(x,y)=\exp(-x)\sum_{n=0}^{\infty}\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\left(T(n,k)\frac{x^k}{k!}\right)\frac{y^n}{n!}$.
Partial differentiation with respect to y gives

$$\begin{aligned}\frac{\partial A}{\partial y} &= \exp(-x)\sum_{n=0}^{\infty}\binom{n}{2}\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!}\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{2n}\left(T(n,k)\frac{x^k}{k!}\right)\frac{y^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{2n+2}\left(T(n+1,k)\frac{x^k}{k!}\right)\frac{y^n}{n!}\,. \end{aligned} \tag{5}$$

Applying the operator $x\frac{d}{dx}$ to $\exp(x)A(x,y)=\sum_{n=0}^{\infty}\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!}$ yields

$$x\exp(x)\left(A+A'\right) = \sum_{n=0}^{\infty}n\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!}, \tag{6}$$

where the prime $'$ indicates differentiation with respect to x.

Apply the operator $x\frac{d}{dx}$ to (6) to find

$$x\exp(x)\left((x+1)A+(2x+1)A'+xA''\right) = \sum_{n=0}^{\infty} n^2\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!}. \quad (7)$$

Subtract (6) from (7) and divide the result by 2 exp(x) to obtain

$$\left(\frac{x^2A+2x^2A'+x^2A''}{2}\right) = \exp(-x) \sum_{n=0}^{\infty}\binom{n}{2}\exp\left(\binom{n}{2}y\right)\frac{x^n}{n!}. \quad (8)$$

The coefficient of $\frac{x^k}{k!}\frac{y^n}{n!}$ on the left of (8) is $\frac{k(k-1)}{2}T(n,k)+2\frac{k(k-1)}{2}T(n,k-1)$ $+\frac{k(k-1)}{2}T(n,k-2)$, while the coefficient of $\frac{x^k}{k!}\frac{y^n}{n!}$ on the right side of (8) equals $T(n+1,k)$ by (5). Hence we have established the recurrence

$$T(n+1,k) = \binom{k}{2}\left(T(n,k)+2T(n,k-1)+T(n,k-2)\right).$$

□

3) A combinatorial interpretation of $T(n,k)$.

The row sums of A122193 begin $[1, 13, 409, 23917, 2244361, ...]$. This is A055203 with the description "the number of different relations between n intervals on a line". There is a link there to a monthly puzzle at IBM research, set by Raul Saavedra - "Imagine you have n events of non-zero duration, in how many different ways could those events overlap in time?" An equivalent formulation of the puzzle is to determine the number of different arrangements of n (nondegenerate) closed intervals on a line. In Saavedra's solution to his problem he sets $W(n,p)$ equal to the number of arrangements of n segments with p endpoints and finds a recurrence for $W(n,p)$. This recurrence turns out to be the same as (4) and with the same initial conditions. Thus we have the following combinatorial interpretation for the entries of A122193: $T(n,k)$ equals the number of arrangements on a line of n (nondegenerate) finite closed intervals having k distinct endpoints.

4) Row polynomials as a black diamond product.

A131689

$$r_n(x) = x \blacklozenge \cdots \blacklozenge x \ (n \text{ factors}) \quad (9)$$

A122193

$$R_n(x) = x^2 \blacklozenge \cdots \blacklozenge x^2 \ (n \text{ factors}) \quad (10)$$

Dukes and White [DuWh'16], in their study of the combinatorics of web diagrams and web matrices, introduced a commutative and associative $\mathbb{C}$-bilinear product of power series, which they named the black diamond product and denoted by the symbol $\blacklozenge$. The black diamond product of monomial polynomials is given by the formula

$$x^m \blacklozenge x^n = \sum_{k=0}^{m} \binom{n+k}{k} \binom{n}{m-k} x^{n+k}. \tag{11}$$

The stated expressions for the row polynomials $r_n(x)$ and $R_n(x)$ as black diamond products may be easily proved by simple induction arguments, making use of the following particular cases of (11):

$$x \blacklozenge x^n = nx^n + (n+1)x^{n+1}$$

and

$$x^2 \blacklozenge x^n = \binom{n}{2} x^n + 2\binom{n+1}{2} x^{n+1} + \binom{n+2}{2} x^{n+2}.$$

5) Formal series expansions of the row polynomials.

A131689

$$r_n(x) = \sum_{i=1}^{\infty} i^n \frac{x^i}{(1+x)^{i+1}}, \quad n \geq 1 \tag{12}$$

$$\frac{1}{1-x} r_n\left(\frac{x}{1-x}\right) = \sum_{i=1}^{\infty} \binom{i}{1}^n x^i, \quad n \geq 1 \tag{13}$$

A122193

$$R_n(x) = \sum_{i=2}^{\infty} \binom{i}{2}^n \frac{x^i}{(1+x)^{i+1}}, \quad n \geq 1 \tag{14}$$

$$\frac{1}{1-x} R_n\left(\frac{x}{1-x}\right) = \sum_{i=2}^{\infty} \binom{i}{2}^n x^i, \quad n \geq 1 \tag{15}$$

Proof of the expansions for $R_n(x)$.

In [Ba'18, section 3.1] we showed that the power series $E_i(x) := \frac{x^i}{(1+x)^i}$, for $i = 0, 1, 2, ...$ form a complete set of orthogonal idempotents in the algebra of formal power series $\mathbb{C}[[x]]$ equipped with the black diamond product. In particular, the idempotents are mutually orthogonal:

$$E_i \blacklozenge E_j \quad = \quad \delta_{ij} E_i \quad i, j \geq 0.$$

Starting from the easily proved idempotent expansion

$$x^2 \quad = \quad \frac{\binom{2}{2}x^2}{(1+x)^3} + \frac{\binom{3}{2}x^3}{(1+x)^4} + \frac{\binom{4}{2}x^4}{(1+x)^5} + \cdots$$

we immediately obtain from (10)

$$\begin{aligned} R_n(x) &= x^2 \blacklozenge \cdots \blacklozenge x^2 \ (n \text{ factors}) \\ &= \frac{\binom{2}{2}^n x^2}{(1+x)^3} + \frac{\binom{3}{2}^n x^3}{(1+x)^4} + \frac{\binom{4}{2}^n x^4}{(1+x)^5} + \cdots, \end{aligned}$$

proving (14). To prove (15) we substitute $\frac{x}{1-x}$ for the variable x in the previous result to give

$$\frac{1}{1-x} R_n\left(\frac{x}{1-x}\right) \quad = \quad \sum_{i=2}^{\infty} \binom{i}{2}^n x^i. \ \square$$

The first few cases of this result are

$$\frac{x^2}{(1-x)^3} \quad = \quad \sum_{i=2}^{\infty} \binom{i}{2} x^i$$

$$\frac{x^2 + 4x^3 + x^4}{(1-x)^5} \quad = \quad \sum_{i=2}^{\infty} \binom{i}{2}^2 x^i$$

$$\frac{x^2 + 20x^3 + 48x^4 + 20x^5 + x^6}{(1-x)^7} \quad = \quad \sum_{i=2}^{\infty} \binom{i}{2}^3 x^i.$$

The numerator polynomials are the row polynomials of A154283.

6) Relationships between the polynomials $r_n(x)$ and $R_n(x)$.

$$R_n(x) \quad = \quad \frac{1}{2^n} \sum_{k=0}^{n} (-1)^{(n-k)} \binom{n}{k} r(n+k, x) \qquad (16)$$

$$R_n(x) \quad = \frac{x}{(1+x)} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} r(n+k, x), \quad n \geq 1 \qquad (17)$$

Proof.

We prove (16), the proof of (17) being exactly similar. From (12), for $n \geq 1$,

$$r_{n+k}(x) = \sum_{i=1}^{\infty} i^{n+k} \frac{x^i}{(1+x)^{i+1}}.$$

Hence

$$\begin{aligned}
\frac{1}{2^n}\sum_{k=0}^{n}(-1)^{(n-k)}\binom{n}{k} r(n+k,x) &= \sum_{i=1}^{\infty}\left(\frac{1}{2^n}\sum_{k=0}^{n}(-1)^{(n-k)}\binom{n}{k} i^k\right) i^n \frac{x^i}{(1+x)^{i+1}} \\
&= \sum_{i=1}^{\infty}\left(\frac{(i-1)^n}{2^n}\right) i^n \frac{x^i}{(1+x)^{i+1}} \\
&= \sum_{i=2}^{\infty}\binom{i}{2}^n \frac{x^i}{(1+x)^{i+1}} \\
&= R_n(x)
\end{aligned}$$

by (14). □

7) Recurrence equation for row polynomials.

A131689

$$r_{n+1}(x) = x\frac{d}{dx}\left((1+x)r_n(x)\right) \qquad (18)$$

A122193

$$R_{n+1}(x) = \tfrac{1}{2}x^2\tfrac{d^2}{dx^2}\left((1+x)^2 R_n(x)\right) \qquad (19)$$

Proof of (19) . From (14)

$$R_n(x) = \sum_{i=2}^{\infty}\binom{i}{2}^n \frac{x^i}{(1+x)^{i+1}}.$$

Therefore

$$(1+x)^2 R_n(x) = \sum_{i=2}^{\infty}\binom{i}{2}^n \frac{x^i}{(1+x)^{i-1}}. \qquad (20)$$

Now one easily checks that

$$\frac{1}{2}x^2\frac{d^2}{dx^2}\left(\frac{x^i}{(1+x)^{i-1}}\right) = \frac{\binom{i}{2}x^i}{(1+x)^{i+1}}. \qquad (21)$$

Hence applying the operator $\frac{1}{2}x^2\frac{d^2}{dx^2}$ to (20) and using (21) we find

$$\begin{aligned} \frac{1}{2}x^2\frac{d^2}{dx^2}\left((1+x)^2R_n(x)\right) &= \sum_{i=2}^{\infty}\binom{i}{2}^{n+1}\frac{x^i}{(1+x)^{i+1}} \\ &= R_{n+1}(x). \square \end{aligned}$$

8) Reflection property for row polynomials.

A131689

$$xr_n(-1-x) = (-1)^n(1+x)r_n(x), \quad n \geq 1 \qquad (22)$$

A122193

$$x^2R_n(-1-x) = (1+x)^2R_n(x), \quad n \geq 1 \qquad (23)$$

Proof of (23).

We assume (22). By (16)

$$R_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}(-1)^{(n-k)}\binom{n}{k}r(n+k,x).$$

Hence

$$\begin{aligned} R_n(-1-x) &= \frac{1}{2^n}\sum_{k=0}^{n}(-1)^{(n-k)}\binom{n}{k}r(n+k,-1-x) \\ &= \frac{1+x}{x}\ \frac{1}{2^n}\sum_{k=0}^{n}(-1)^{(n-k)}(-1)^{n+k}\binom{n}{k}r(n+k,x)\ [n \geq 1] \\ &= \frac{(1+x)^2}{x^2}R_n(x) \end{aligned}$$

for $n \geq 1$ by (17). $\square$

9) Worpitzky-type identities.

A131689

$$\sum_{k=1}^{n}t(n,k)\binom{x}{k} = \binom{x}{1}^n \qquad (24)$$

A122193

$$\sum_{k=2}^{2n}T(n,k)\binom{x}{k} = \binom{x}{2}^n \qquad (25)$$

Proof of (25).

The proof is by induction. The base case $n = 1$ is trivial. Suppose the identity holds for some $n > 1$, then by the recurrence relation (4),

$$\begin{aligned}
\sum_{k=2}^{2n+2} T(n+1,k)\binom{x}{k} &= \sum_{k=2}^{2n+2} \left(\binom{k}{2}(T(n,k)+2T(n,k-1)+T(n,k-2))\right)\binom{x}{k} \\
&= \sum_{k=2}^{2n+2} T(n,k)\binom{k}{2}\binom{x}{k} + 2\sum_{k=2}^{2n+2} T(n,k-1)\binom{k}{2}\binom{x}{k} \\
&\quad + \sum_{k=2}^{2n+2} T(n,k-2)\binom{k}{2}\binom{x}{k} \\
&= \sum_{k=2}^{2n+2} T(n,k)\binom{k}{2}\binom{x}{k} + 2\sum_{k=1}^{2n+1} T(n,k)\binom{k+1}{2}\binom{x}{k+1} \\
&\quad + \sum_{k=0}^{2n} T(n,k)\binom{k+2}{2}\binom{x}{k+2} \\
&= \sum_{k=2}^{2n} T(n,k)\left(\binom{k}{2}\binom{x}{k} + 2\binom{k+1}{2}\binom{x}{k+1} + \binom{k+2}{2}\binom{x}{k+2}\right) \\
&= \sum_{k=2}^{2n+2} T(n,k)\binom{x}{2}\binom{x}{k} \\
&= \binom{x}{2}^{n+1}
\end{aligned}$$

by the induction hypothesis, and the proof by induction goes through. □

10) Finite power sums.

A131689

$$\sum_{i=1}^{n-1} i^m = \sum_{k=1}^{m} t(m,k)\binom{n}{k+1} \qquad m \geq 1 \tag{26}$$

A122193

$$\sum_{i=1}^{n-1} \binom{i}{2}^m = \sum_{k=2}^{2m} T(m,k)\binom{n}{k+1} \qquad m \geq 1 \tag{27}$$

Proof of (27). Follows immediately from (25) and the hockey-stick identity for binomial coefficients

$$\sum_{i=k}^{N} \binom{i}{k} = \binom{N+1}{k+1}. \; \square$$

11) Generalisations

It is clear that the above results can be extended to arrays with a double e.g.f. of the form

$$\exp(-x)\sum_{n=0}^{\infty} \exp\left(\binom{n}{m} y\right) \frac{x^n}{n!}$$

for $m = 3, 4, ...$ For example, when $m = 3$ the table begins

$n\diagdown k$	3	4	5	6	7	8	9	10	11	12
1	1									
2	1	12	30	20						
3	1	60	690	2940	5670	5040	1680			
4	1	252	8730	103820	581700	1767360	3087000	3099600	1663200	369600
...										

This table is currently not in the OEIS. We denote the table entries by $\widetilde{T}(n,k)$ and the row poynomials by $\widetilde{R}(n,x)$. We summarise some of the properties of this table below.

$$\widetilde{T}(n,k) = \sum_{i=0}^{k}(-1)^{(k-i)}\binom{k}{i}\binom{i}{3}^n$$

$$T(n+1,k) = \binom{k}{3}\left(T(n,k)+3T(n,k-1)+3T(n,k-2)+T(n,k-3)\right)$$

$$\widetilde{R}(n,x) = x^3 \blacklozenge \cdots \blacklozenge x^3 \text{ (n factors)}$$

$$\widetilde{R}(n+1,x) = \tfrac{1}{3!}x^3\tfrac{d^3}{dx^3}\Big((1+x)^3\widetilde{R}(n,x)\Big)$$

$$x^3\widetilde{R}(n,-1-x) = (-1)^n(1+x)^3\widetilde{R}(n,x) \quad [n \geq 1]$$

$$\widetilde{R}(n,x) = \sum_{k=3}^{\infty}\binom{k}{3}^n\frac{x^k}{(1+x)^{k+1}} \quad [n \geq 1]$$

$$\sum_{k=3}^{3n}\widetilde{T}(n,k)\binom{x}{k} = \binom{x}{3}^n$$

$$\sum_{i=1}^{n-1}\binom{i}{3}^m = \sum_{k=3}^{3m}\widetilde{T}(m,k)\binom{n}{k+1} \quad [m \geq 1]$$

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