Recurrence relation for nonisomorphic connected k-multigraphs

These consist of either no edge between vertices or one of k types of distinguishable undirected edges. Harary and Palmer in *Graphical Enumeration* page 52 tell us the connectedness relation between the OGFs of a class of graphs $\mathcal{G}$ and the subclass of connected graphs of this type $\mathcal{C}$.

$$G(z) = \exp \sum_{\ell \ge 1} \frac{C(z^\ell)}{\ell}$$

This is the unlabeled multiset operator from *Analytic Combinatorics* by Flajolet and Sedgewick. Differentiate to obtain

$$G'(z) = \exp \sum_{\ell \ge 1} \frac{C(z^\ell)}{\ell} \sum_{\ell \ge 1} C'(z^\ell) z^{\ell-1} \\ = G(z) \sum_{\ell \ge 1} C'(z^\ell) z^{\ell-1}.$$

Extracting coefficients we find

$$[z^n]G'(z) = (n+1)G_{n+1} = \sum_{q=0}^{n} [z^{n-q}]G(z)[z^q] \sum_{\ell \ge 1} C'(z^\ell) z^{\ell-1} \\ = \sum_{q=0}^{n} G_{n-q} [z^{q+1}] \sum_{\ell \ge 1} C'(z^\ell) z^\ell = \sum_{q=0}^{n} G_{n-q} \sum_{\ell | q+1} [z^{\ell \times (q+1)/\ell}] C'(z^\ell) z^\ell \\ = \sum_{q=0}^{n} G_{n-q} \sum_{\ell | q+1} [z^{(q+1)/\ell}] C'(z) z = \sum_{q=0}^{n} G_{n-q} \sum_{\ell | q+1} \frac{q+1}{\ell} C_{(q+1)/\ell} \\ = \sum_{q=0}^{n} G_{n-q} \sum_{p | q+1} p C_p.$$

This gives a recurrence

$$(n+1)G_{n+1} = \sum_{q=0}^{n-1} G_{n-q} \sum_{p|q+1} pC_p + \sum_{p|n+1 \wedge p<n+1} pC_p + (n+1)C_{n+1}$$

or

$$C_{n+1} = G_{n+1} - \frac{1}{n+1} \sum_{p|n+1 \wedge p<n+1} pC_p - \frac{1}{n+1} \sum_{q=0}^{n-1} G_{n-q} \sum_{p|q+1} pC_p.$$

We simplify to

$$C_n = G_n - \frac{1}{n} \sum_{p|n \wedge p<n} pC_p - \frac{1}{n} \sum_{q=0}^{n-2} G_{n-1-q} \sum_{p|q+1} pC_p$$

which is

$$\boxed{C_n = G_n - \frac{1}{n} \sum_{p|n \wedge p<n} pC_p - \frac{1}{n} \sum_{q=1}^{n-1} G_{n-q} \sum_{p|q} pC_p.}$$