The Number of Binary Cube-Free Words of length up to 47 and Their Numerical Analysis

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Abstract

An analog of the work of Noonan and Zeilberger on square-free ternary words is given for the case of cube-free binary words. As in their work, the Goulden-Jackson Cluster Method is used to derive a rigorous upper bound, as well as a non-rigorous estimate, for the limit of the n^{th} roots of the terms. The Maple implementation of this work is available from this paper's website **http://www.math.temple.edu/~anne/cubefree.html**.

Preface

In order to define a cube-free word we must first define a factor of a word. Given a word $w = w_1w_2 \ldots w_n$, a *factor* of w is any subword of the form $w_k w_{k+1} \ldots w_{k+j}$ where $1 \leq k \leq n$ and $0 \leq j \leq n-1$. A word is *cube-free* if it contains no factors of the form xxx, where x is any word. My Maple package Cubefree (available from **http://www.math.temple.edu/~anne/cubefree.html**) can be used to derive cube-free words on any given alphabet up to the required length. The number of binary cube-free words of length at most n for $0 \leq n \leq 47$ are given below.

The Sequence of Binary Cube-Free Words of length up to 47

1, 2, 4, 6, 10, 16, 24, 36, 56, 80, 118, 174, 254, 378, 554, 802, 1168, 1716, 2502, 3650, 5324, 7754, 11320, 16502, 24054, 35058, 51144, 74540, 108664, 158372, 230800, 336480, 490458, 714856, 1041910, 1518840, 2213868, 3226896, 4703372, 6855388, 9992596, 14565048, 21229606, 30943516, 45102942, 65741224, 95822908, 139669094.

The 'Connective Constant'

Let a_n denote the number of cube-free binary words of length n. The values of a_n for $n = 0, \ldots, 47$ are given above. It is well-known, and easy to see, that $\mu := \lim_{n\to\infty} a_n^{1/n}$ exists. Using the 'memory-45' analog (i.e. the corresponding sequence that enumerates words that avoid cubes x^3, with $length(x) \leq 15$), that was generated using the Maple package, up to word-length 300, we find the rigorous upper bound $\mu < 1.457579200596766$.

Using Zinn's method, we also found that, assuming that $a_n \sim n^\theta \mu^n$, then $\mu \approx 1.457$, and $\theta \approx 0$. Hence it is reasonable to conjecture that $a_n \sim \mu^n$, where $\mu := \lim_{n \to \infty} a_n^{1/n} \approx 1.457$.

References

[1] John Noonan and Doron Zeilberger, *The Goulden-Jackson Cluster Method: Extensions, Applications and Implementations*, preprint. Available from the paper's website **http://www.math.temple.edu/~zeilberg/gj.html**.