Prandtl-Blasius Flow

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The boundary value problem

$$y'''(x) + y''(x)y(x) = 0, \qquad y(0) = 0, \qquad y'(0) = 0, \qquad \lim_{x\to\infty} y'(x) = 1$$

arises in the study of two-dimensional incompressible viscous flow past a thin semi-infinite flat plate [1, 2, 3, 4]. Such an equation is similar to the Thomas-Fermi equation [5], but is even more difficult to solve (because it is of higher order).

A well-known series for $y(x)$ is

$$y(x) = \sum_{k=0}^{\infty} (-1)^k \frac{p_k \xi^{k+1}}{(3k+2)!} x^{3k+2}, \qquad x \approx 0$$

where $p_0 = 1$ and [6]

$$p_k = \sum_{j=0}^{k-1} \binom{3k-1}{3j} p_j p_{k-j-1}, \qquad k \geq 1.$$

Hence it is important to compute

$$\xi = \lim_{x\to 0^+} \frac{y(x) - 0}{x^2/2} = y''(0)$$

as accurately as feasible. Blasius' series has only a finite radius of convergence [7, 8, 9, 10, 11]:

$$\rho = \lim_{k\to\infty} \left(\frac{(3k)(3k+1)(3k+2)p_{k-1}}{p_k \xi} \right)^{1/3} = 4.0234644935...$$

(in fact, the associated singularities are at $x = -\rho$ and $\rho \exp(\pm i\pi/3)$). Unlike the Thomas-Fermi equation, an efficient transformation for the Blasius equation is not yet known that permits high-precision estimates of ξ. A Runge-Kutta numerical ODE solver gives $\xi = 0.4695999883...$, as well as [2, 3, 10, 11]

$$\eta = \lim_{x\to\infty} (x - y(x)) = 1.2167806216....$$

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The fluid dynamics literature is somewhat bewildering because of (small) variations in the presentation of Blasius' equation. Let us generalize our discussion to clear up any confusion. Consider

$$z'''(x) + a\,z''(x)z(x) = 0, \qquad z(0) = 0, \qquad z'(0) = 0, \qquad \lim_{x\to\infty} z'(x) = b$$

where $a > 0$, $b > 0$. Let $c = z''(0)$; it can be easily shown that $c = a^{1/2}b^{3/2}\xi$ and thus

$$c(a = 1/2, b = 1) = \xi/\sqrt{2} = 0.3320573362...,$$

$$c(a = 1, b = 2) = 2\sqrt{2}\xi = 1.3282293448... = 2(0.6641146724...),$$

$$b(a = 1, c = 1) = \xi^{-2/3} = 1.6551903602...$$

$$b(a = 1/2, c = 1) = 2^{1/3}\xi^{-2/3} = 2.0854091764....$$

From formulas for the radius of convergence

$$R = \frac{\rho}{(a\,b)^{1/2}} = \left(\frac{\xi}{a\,c}\right)^{1/3}\rho$$

and for the limit

$$L = \lim_{x\to\infty}(b\,x - z(x)) = \left(\frac{b}{a}\right)^{1/2}\eta,$$

we obtain

$$R(a = 1/2, b = 1) = \sqrt{2}\rho = 5.6900380545...,$$

$$R(a = 1, c = 1) = \xi^{1/3}\rho = 3.1273479155...,$$

$$R(a = 2, c = 1) = (\xi/2)^{1/3}\rho = 2.4821776854...$$

(long ago Weyl [12, 13] gave bounds 2.08 and 3.11 for the latter) and

$$L(a = 1, b = 2) = L(a = 1/2, b = 1) = \sqrt{2}\eta = 1.7207876575...,$$

$$L(a = 2, b = 1) = L(a = 1, b = 1/2) = \eta/\sqrt{2} = 0.8603938287....$$

When moving fluid encounters a solid, a layer is formed adjacent to the boundary of the solid. Strong frictional effects exist inside this layer; on the outside, by contrast, the flow essentially displays no viscosity [2, 14]. For the case of a thin plate, the fluid velocity changes rapidly from zero (along the plate) to its original value (beyond the boundary layer). Three relevant quantities in this physical model are the *displacement thickness*

$$\delta_1 = \int_0^\infty (1 - y'(x))\,dx = \eta = 1.2167806216...,$$

the *momentum thickness*

$$\delta_2 = \int_0^\infty y'(x)\,(1 - y'(x))\,dx = \xi = 0.4695999883...$$

and the *energy thickness*

$$\delta_3 = \int_0^\infty y'(x)\left(1 - y'(x)^2\right)dx = 2\int_0^\infty y''(x)y'(x)y(x)dx = 0.73848498....$$

It is also known that [2, 10]

$$y''(x) \sim \kappa \exp\left[-(x-\eta)^2/2\right]$$

as $x \to \infty$, where $\kappa = 0.3305407719... = (0.2337276212...)\sqrt{2}$. We wonder whether δ_3 and κ are closely related. The literature associated with $y(x)$ is massive [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54].

0.1. Falkner-Skan Equation. Consider

$$y'''(x) + y''(x)y(x) + \lambda\,(1 - y'(x)^2) = 0, \qquad y(0) = 0, \qquad y'(0) = 0, \qquad \lim_{x\to\infty} y'(x) = 1$$

which arises in the study of viscous flow past a wedge of angle $\lambda\,\pi$, $0 \leq \lambda \leq 1$. The special case $\lambda = 0$ is Blasius' equation, in which the wedge reduces to a flat plate. The special case $\lambda = 1/2$ is called Homann's equation; we here have [2, 3, 55, 56, 57]

$$y''(0) = 0.92768003... = (1.31193769...)/\sqrt{2}, \qquad \lim_{x\to\infty}(x - y(x)) = 0.804548....$$

The special case $\lambda = 1$ is called Hiemenz's equation (corresponding to stagnation flow, for example, past a large disk); we here have [2, 3, 58, 59]

$$y''(0) = 1.23258765..., \qquad \lim_{x\to\infty}(x - y(x)) = 0.647900....$$

It is known that a smooth solution $y(x)$ exists and is unique [12, 13, 60, 61, 62, 63, 64, 65] for each λ, $0 \leq \lambda \leq 1$. An especially simple proof for $\lambda = 0$, due to Serrin, appears in [66, 67].

Physically relevant solutions also exist for negative λ, more precisely, in the range $-0.19883768... = \mu \leq \lambda < 0$. (Positive λ corresponds to flow toward the wedge; negative λ corresponds to flow away from the wedge.) By "physically relevant", we mean that a solution $y(x)$ further satisfies

$$0 < y'(x) < 1 \qquad \text{for all } x > 0,$$

$$1 - y'(x) = O\left(e^{-\gamma x}\right) \qquad \text{as } x \to \infty$$

for some $\gamma > 0$. It follows that $y''(0) > 0$ when $\lambda > \mu$ and $y''(0) = 0$ when $\lambda = \mu$. A deeper understanding of the constant μ is desired [17, 58, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79]. Again, the associated literature is massive [18, 19, 20, 21, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102].

0.2. Streamlines. At each point in the first quadrant of (s, t)-space, define a velocity vector (u, v) by

$$u(s,t) = s^m y'(\theta),$$

$$v(s,t) = -\sqrt{\frac{s^{m-1}}{2(2-\lambda)}}\left((\lambda - 1)\theta\, y'(\theta) + y(\theta)\right)$$

where

$$m = \frac{\lambda}{2-\lambda}, \qquad \theta = t\sqrt{(m+1)s^{m-1}}.$$

The vector field $(s, t) \mapsto (u, v)$ determines the streamlines for laminar boundary-layer fluid flow past a wedge, as suggested in the Figure, for a specified viscosity coefficient [3, 83].

0.3. Addendum. Recent papers devoted to the Blasius ODE include [103, 104, 105]; related examples (with numerical estimates) are discussed in [106, 107].

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