A16825 A-110185 A-165355 A-5131 A-9317-8 A-58281

CONTINUED FRACTIONS CONTAINING ARITHMETIC PROGRESSIONS

By D. H. LEHMER

THE remarkable regular continued fractions

(1)
$$2 + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \frac{1}{18} + \dots = (e+1)/(e-1) = \coth(\frac{1}{2}),$$

A16825

(2)
$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots = (e^2 + 1)/(e^2 - 1) = \coth(1),$$

and the general case

$$2S + \frac{1}{6S} + \frac{1}{10S} + \frac{1}{14S} + \dots = (e^t + 1)/(e^t - 1) = \coth(t/2),$$

where t = 1/S, were discovered by Euler [1]. They are examples of elegant continued fractions with elegant values. In looking for further examples one can ask for the values of other continued fractions whose partial quotients are in arithmetical progression, hoping that these values may also be elegant, or at least simple, in some sense.

The first general task is that of evaluating the continued fraction whose partial quotients form a general arithmetic progression.

(3)
$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots = [a_0, a_1, a_2, a_3, \dots],$$

where $a_n = an + b \ (a \neq 0)$.

We denote this fraction by F(a, b).

Its value is given by

THEOREM 1

$$F(a, b) = I_{\alpha-1}(2a^{-1})/I_{\alpha}(2a^{-1}),$$

where $I_{\nu}(z)$ is the modified Bessel function [2],

(4)
$$I_{\nu}(z) = i^{-\nu} J_{\nu}(iz) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{\Gamma(m+1)\Gamma(\nu+m+1)},$$

and where $\alpha = b/a$.

Received July 20, 1970.

Proof: The reason why Bessel functions are involved in this problem is because of the familiar recurrence, which follows easily from (4),

$$I_{\nu-1}(z) = \frac{2\nu}{z} I_{\nu}(z) + I_{\nu+1}(z).$$

If we set

$$m = n + \alpha = n + b/a$$

and

$$z=2a^{-1},$$

we see that the function

$$U_n = U_n(a, \alpha) = I_{n+\alpha}(2a^{-1})$$

enjoys the recurrence

$$U_{n-1} = (an + b)U_n + U_{n+1}$$
.

Starting with n = 0, we can write

$$U_{-1} = bU_0 + U_1 = a_0U_0 + U_1,$$

 $U_0 = (a + b)U_1 + U_2 = a_1U_1 + U_2,$

and

$$U_1 = (2a + b)U_2 + U_3 = a_2U_2 + U_3.$$

Eliminating U_1 , U_2 , U_3 , ..., we find

$$U_{-1}/U_0 = a_0 + U_1/U_0$$

$$= a_0 + \frac{1}{U_0/U_1}$$

$$= a_0 + \frac{1}{a_1} + \frac{1}{U_1/U_2}$$

$$= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_k} + \frac{1}{U_k/U_{k+1}}$$

Now as $k \to \infty$

$$\frac{U_k}{U_{k+1}} = \frac{I_{k+\alpha}(2a^{-1})}{I_{k+\alpha+1}(2a^{-1})} \sim a(k+\alpha+1) \to \infty,$$

as we see from the expansion (4).

Hence the infinite continued fraction F(a, b) converges to

$$U_{-1}/U_0 = I_{\alpha-1}(2a^{-1})/I_{\alpha}(2a^{-1}),$$

proving the theorem.

When ν is half an odd integer, $I_{\nu}(z)$ is an elementary function. In particular

$$I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} z^{-\frac{1}{2}} \cosh z$$

and

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} z^{-\frac{1}{2}} \sinh z.$$

Euler's examples now are seen to be the cases F(4, 2) for (1), F(2, 1) for (2), and F(4S, 2S) for (3). In all three cases $\alpha = \frac{1}{2}$, which leads to the hyperbolic cotangent $I_{-\frac{1}{2}}(z)/I_{\frac{1}{2}}(z)$.

We can now evaluate the most natural continued fraction of our class,

$$F(1, 1) = [1, 2, 3, 4, 5, \ldots] = I_0(2)/I_1(2),$$

in terms of the "imaginary" Bessel function. If one insists on the "real" Bessel function $J_{\nu}(z)$, one can have it at the price of using the barely irregular continued fraction whose numerators are all equal to -1. Thus

Theorem 2. The irregular continued fraction

$$b-\frac{1}{a+b}-\frac{1}{2a+b}-\frac{1}{3a+b}-\dots=J_{\alpha-1}(2a^{-1})/J_{\alpha}(2a^{-1}),$$

where, as before, $\alpha = b/a$.

This follows from Theorem 1 by simply replacing a and b by ia and ib. Thus we have evaluated

(5)
$$F_{\epsilon}(a,b) = b + \frac{\epsilon}{a+b} + \frac{\epsilon}{2a+b} + \frac{\epsilon}{3a+b} + \cdots$$
$$= \begin{cases} I_{\alpha-1}(2a^{-1})/I_{\alpha}(2a^{-1}) & \text{if } \epsilon = 1\\ J_{\alpha-1}(2a^{-1})/J_{\alpha}(2a^{-1}) & \text{if } \epsilon = -1. \end{cases}$$

Theorem 1 can be used to evaluate continued fractions having two interlaced arithmetic progressions as their partial quotients. For example, we have

$$[1, 12, 5, 28, 9, 44, 13, 60, 17, 76, \ldots] = (\frac{1}{2})(e+1)/(e-1) = \frac{1}{2} \coth(\frac{1}{2})$$

(6) $(2, 7, 5, 13, 8, 19, 11, 25, 14, 31, 17, ...] = (\sqrt{2}/2)I_{\frac{1}{2}}(2\sqrt{2}/3)/I_{\frac{1}{2}}(2\sqrt{2}/3).$ A / 65355

The general theorem here is

THEOREM 3. Let a, b, c, d be integers satisfying ([3])

$$(7) 2bc = d(2a+b)$$

Then the continued fraction

(8)
$$[a, c, a + b, c + d, a + 2b, c + 2d, a + 3b, c + 3d, \ldots]$$

= $m^{-1}I_{\alpha-1}(2|l)/I_{\alpha}(2|l)$,
where $m = \sqrt{d/b}$, $l = \frac{1}{2}\sqrt{bd}$, $\alpha = 2a/b$.

Proof: Let k = am. Then for every nonnegative integer h,

$$(9) m(a+hb)=k+2lh,$$

while

$$\frac{1}{m}(c + hd) = \sqrt{\frac{b}{d}}(c + hd) = \frac{1}{2}\frac{2bc}{\sqrt{bd}} + h\sqrt{bd}$$

$$= \frac{1}{2}\frac{d(2a + b)}{\sqrt{bd}} + 2hl$$

$$= \frac{1}{2}m(2a + b) + 2hl$$

$$= \frac{1}{2}(ma + m(a + b)) + 2hl$$

$$= \frac{1}{2}(k + k + 2l) + 2hl$$

$$= k + (2l + 1)h,$$

where we have used (7) and (9). Hence the continued fraction

(10)
$$[ma, c/m, m(a + b), (c + d)/m), \ldots] = [k, k + l, k + 2l, k + 3l, \cdots]$$

= $F(l, k)$
= $I_{\alpha-1}(2l^{-1})/I_{\alpha}(2l^{-1}),$

where $\alpha = k/l = 2am/\sqrt{db} = 2a/b$.

But this fraction is clearly m times the fraction (8).

This proves the theorem.

The fraction (10) to which we have applied Theorem 1 is, to be sure, not necessarily a regular continued fraction since its partial quotients need not be integers nor even rational. However, in the Proof of Theorem 1 we did not use the hypothesis that a and b were integers. Thus, in the fraction (6) we have

$$a = 2$$
, $b = 3$, $c = 7$, $d = 6$,

so that

$$m = \sqrt{2}, \quad l = \frac{3}{2}\sqrt{2}, \quad k = 2\sqrt{2}.$$

Theorem 1 can also be used to evaluate a class of continued fractions in which the partial quotients in arithmetic progression are separated by constant strings of arbitrary partial quotients, that is, fractions like

(11)
$$[b, c_1, c_2, \ldots, c_{k-1}, a+b, c_1, c_2, \ldots, c_{k-1}, 2a+b, c_1, \cdots].$$

Fractions of this type arise naturally, as Euler discovered. For example, the regular continued fraction for e-1 is

$$e-1=[1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 10, 1, 1, 14, \ldots].$$

It is clear that we can restrict our attention to continued fractions of the form (11), since this is equivalent to (in the sense of being a linear fractional function of) any other continued fraction which is obtained by prefixing any finite set of arbitrary partial quotients.

THEOREM 4. The continued fraction (11) is equal to

$$\frac{1}{B}\{-B' + \epsilon(aB, bB + B' + B'')\},\$$

where

$$[b, c_1, c_2, \dots, c_{k-1}] = A/B$$

$$[b, c_1, c_2, \dots, c_{k-2}] = A'/B'$$

$$[c_1, c_2, c_3, \dots, c_{k-1}] = A''/B''$$

and

$$\epsilon = (-1)^{k-1}.$$

Proof: This follows from the application of the so-called contraction operator [see Perron [4], p. 202, formula 9]. In fact the continued fraction (11) has the value

$$\frac{1}{B}\left\{bB+B''+\frac{|(-1)^{k-1}|}{(a+b)B+B'+B''}+\frac{|(-1)^{k-1}|}{(2a+b)B+B'+B''}+\cdots\right\}$$

where we now have a continued fraction of type (5) after adding B' to its first term.

For example, consider

$$[1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, 1, \ldots] = \frac{\cos(1)}{\sin(1) - \cos(1)}$$

Here we have

$$b=1, \quad a=2, \quad k=2, \quad b=1, \quad b'=1, \quad B''=1, \quad \epsilon=-1,$$

$$\alpha=\frac{bB+B'+B''}{aB}=\frac{3}{2}.$$

Hence the fraction is equal to

$$-1 + F_{-1}(2,3) = -1 + \frac{J_{\frac{1}{2}}(1)}{J_{\frac{3}{2}}(1)} = \frac{\cos(1)}{\sin(1) - \cos(1)}$$

If we prefix this continued fraction by 1, we get the regular continued fraction for

$$1 + \frac{\sin{(1)} - \cos{(1)}}{\cos{(1)}} = \tan{(1)}.$$

A93178
with diffe

Another example whose derivation is left to the reader is Euler's

$$[1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, 1, 1, 17, 1, \ldots] = \sqrt{e}.$$
 A5828

Theorem 1 may be used to furnish facts about Bessel functions.

An example is

THEOREM 5. If c is a constant,

$$\lim_{\alpha = \infty} I_{\alpha - 1}(c\alpha)/I_{\alpha}(c\alpha) = \exp\left(\operatorname{arcsinh}(c^{-1})\right) = c^{-1}(1 + \sqrt{1 + c^2}).$$

Proof: Let b = 2/c and $a = 2/(c\alpha)$.

Then by Theorem 1

$$F(a, b) = [b, a + b, 2a + b, \ldots] = I_{\alpha-1}(c\alpha)/I_{\alpha}(c\alpha),$$

as $\alpha \to \infty$, $a \to 0$, and we get, in the limit,

$$\lim_{\alpha\to\infty}\frac{I_{\alpha-1}(c\alpha)}{I_{\alpha}(c\alpha)}=F(0,b)=[b,b,b,\ldots].$$

But F(0, b) is the positive root of the quadratic equation

$$b+\frac{1}{x}=x.$$

Therefore,

$$F(0, b) = \frac{1}{2}(b + \sqrt{1 + b^2}) = c^{-1}(1 + \sqrt{1 + c^2}).$$

This proves the theorem.

For those readers who do not feel comfortable about passing to the limit termwise we give an alternate proof. For a more direct proof we need a formula for $I_{\nu}(z)$ which is valid for large ν and large z. The power series (4), of course, will not do. Instead we can use [5]

(12)
$$I_{\nu}(\nu x) = \frac{1}{\sqrt{2\pi\nu z}} \left\{ \frac{z-1}{x} e^{z} \right\}^{\nu} (1 + 0(\nu^{-1})),$$

where, for simplicity, we have let

$$z=\sqrt{1+x^2}.$$

To apply this asymptotic expression to the proof of Theorem 5, we set

$$v_1 = \alpha$$
, $x_1 = c$, $z_1 = \sqrt{1 + c^2} = d$

$$v_2 = \alpha - 1$$
, $x_2 = \frac{c\alpha}{\alpha - 1}$, $x_2 = \sqrt{1 + x_2^2}$.

Then

$$\frac{I_{\nu_2}(\nu_2 x_2)}{I_{\nu_1}(\nu_1 x_1)} = \frac{I_{\alpha-1}(c\alpha)}{I_{\alpha}(c\alpha)}.$$

By (12) we have

$$\frac{I_{\nu_2}(\nu_2 x_2)}{I_{\nu_1}(\nu_1 x_1)} \sim \left(\frac{\nu_2 z_2}{\nu_1 z_1}\right)^{\nu_2} \left(\frac{(z_2 - 1) x_1 e^{z_2 - z_1}}{(z_1 - 1) x_2}\right)^{\alpha} \left(\frac{z_2 - 1}{x_2} e^{z_2}\right)^{-1}.$$

As ν_1 and $\nu_2 = \nu_1 - 1$ tend to infinity, the first factor tends to 1. The last factor tends to $ce^{-d}/(d-1)$.

Since

$$\frac{c}{d-1}=\frac{c}{\sqrt{1+c^2}-1}=c^{-1}(1+\sqrt{1+c^2}),$$

it suffices to show that the middle factor tends to e^{d} .

To this effect we let $\epsilon = \alpha^{-1}$ and ignore terms in ϵ^2 in what follows. First we find

$$x_2 = \frac{c\alpha}{\alpha - 1} = c \frac{1}{1 - \epsilon} = c(1 + \epsilon);$$

next,

$$z_2 = \sqrt{1 + x_2^2} = d\left(1 + \frac{c^2 \epsilon}{d^2}\right),$$
 $z_2 - 1 = (d - 1)\left(1 + \frac{c^2 \epsilon}{d(d - 1)}\right),$
 $z_2 - z_1 = c^2 \epsilon / d.$

Hence,

$$\frac{(z_2 - 1)x_1}{(z_1 - 1)x_2} e^{z_2 - z_1} = 1 + \left(\frac{c^2}{d(d - 1)} - 1 + \frac{c^2}{d}\right) \epsilon$$
$$= 1 + d\epsilon.$$

It is now evident that the middle factor tends to

$$\lim_{\epsilon \to 0} (1 + d\epsilon)^{1/\epsilon} = e^d.$$

The continued fraction (3) belongs to a much wider class of continued fractions associated with the name of Hurwitz (see Perron [4], p. 126). The convergents of (3) have interesting congruential properties (see [6] and [7]).

REFERENCES

1. L. Euler, "De fractionibus continuis dissertatio", Comm. Acad. Sci. Petropolitanae, 9 (1737), 98-137, Opera Omnia, Leipzig (1925), S.1, 14, 186-215.

2. See for example G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge (1922).

3. This diophantine equation may be solved by choosing b and c arbitrary integers and then taking for d any divisor of 2bc whose codivisor is of the same parity as b.

4. O. Perron, Die Lehre von den Kettenbrüchen, Berlin (1913), Leipzig (1929).

5. D. H. Lehmer, "Note on the Computation of the Bessel Function $I_n(x)$ ", Math. Tables and Aids to Computation (now Math. Comp.), 1 (1944), 133-135.

6. D. N. Lehmer, "Arithmetical Properties of Certain Hurwitzian Continued Fractions",

4. O. Perron, Die Laine van den Mattanbert, Berlin (1913), Leinnig (1929).

Amer. Jour. Math., 40 (1918), 375-390.
7. D. H. Lehmer, "Arithmetical periodicities of Bessel Functions", Annals of Math., s. 2, **33** (1932), 143–150.

UNIVERSITY OF CALIFORNIA BERKELEY