## UNDERSTANDING THE FACE STRUCTURE OF THE KUNZ CONE INTENDED FOR THE DEPARTMENT OF MATHEMATICS AND STATISTICS

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by Emily Irene O'Sullivan Spring 2023

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Intended For The

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#### ABSTRACT OF THE THESIS

Understanding the Face Structure of the Kunz Cone Intended For The Department of Mathematics and Statistics

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The Kunz cone is a geometric object defined by a set of inequalities of the form  $x_i + x_j \ge x_{i+j}$  where subscripts are in  $\mathbb{Z}_m$ . Setting some subset of those inequalities equal, we obtain a face of the cone. In this thesis, we will explore the face structures of the 4-, 5-, and 6-dimensional cones by creating 3-dimensional projections of each. We will also characterize the faces of codimension 2 for any Kunz cone.

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## CHAPTER 1 Introduction

## The Kunz cones are a family of polyhedra first studied in the 1980s by Ernst Kunz [5]. They are defined by a set of inequalities of the form $x_i + x_j \ge x_{i+j}$ where subscripts are in $\mathbb{Z}_m$ . Interest has been renewed in the family of Kunz cones in recent years, and many recent papers [1, 3, 4] have studied various aspects of the Kunz cones and its connections to numerical semigroups [6]. This thesis takes a more geometric approach to studying the Kunz cone, focusing on face structure instead of numerical semigroups. We begin in Chapter 2 with an introduction to polyhedral geometry more generally [7], and then proceed into geometry specific to the Kunz cone.

A face of the cone can be obtained with some subset of the defining inequalities set to equality. Faces range in dimension from 0 to the full dimension of the cone. In Chapter 3 we explore in detail the full face structures of the 4-, 5-, and 6-dimensional Kunz cones using projection techniques to obtain 3-dimensional models for easier visualization. In Chapter 4 we study the Kunz cones more generally to determine how to predict when two faces of codimension 1 intersect in a face of codimension 2 and when not. Finally, in Chapter 5 we look towards future questions that could build on this research.

SageMath code used for this thesis is posted at https://github.com/emerflee/Kunz-Cone.

#### CHAPTER 2

#### Background

We will begin with defining some relevant polyhedral geometry and then build to the definition of the family of Kunz cones. Next, we will touch on posets and projections as two tools for better understanding and identifying the face structure of the Kunz cones that will be studied in greater detail in Chapter 3.

#### 2.1 Polyhedral Geometry

In this section, we introduce some polyhedral geometry. First, a few definitions; then, some examples.

**Definition 2.1.** A *halfspace* is the set of solutions to a linear inequality

$$a_1 x_1 + a_2 x_2 + \dots + a_d x_d \ge b. \tag{2.1}$$

The boundary of a halfspace (the set of points which satisfy the above inequality with equality) is called a *hyperplane*. A *polyhedron* (plural: polyhedra) is the intersection of finitely many halfspaces. Note that this means every halfspace is a polyhedron. A *cone* is a polyhedron in which all of the hyperplanes have the origin as a solution. This means that b = 0 in the linear inequality (2.1). A cone is *pointed* if it contains no linear subspaces.

**Example 2.2.** Refer to Figure 2.1. The graph of  $x_1 + x_2 \ge 2$  in 2.1a is a halfspace and a polyhedron. It is not a cone because the origin is not a solution of its hyperplane, the line  $x_1 + x_2 = 2$ . The graph of  $2x_1 - x_2 \ge 0$  in 2.1b is also a halfspace and a polyhedron, but because the origin is a solution to its hyperplane  $2x_1 - x_2 = 0$ , it is also a cone. It is not, however, a pointed cone, because the line  $2x_1 - x_2 = 0$  is contained in its entirety. Next, we intersect each of these with the halfspace  $-x_1 + 2x_2 \ge 0$ . In 2.1c, we have a polyhedron, but not a cone (since the origin is not a solution of  $x_1 + x_2 = 2$ ). In 2.1d, we have a cone since both hyperplanes have the origin as a solution. This cone is a pointed cone because no lines are contained in it – notice the angle at the origin is less than 180° so containment of a line is impossible. In summary, (a) and (b) are halfspaces, (b) and (d) are cones, (d) is a pointed cone, and all are polyhedra.

A common way to describe a polyhedron is with a list of irredundant inequalities, each of which defines a halfspace. This is called an H-description and can be written in matrix form.



Figure 2.1. Examples of halfspaces, polyhedra, and cones.

The H-description for Figure 2.1c is below, first in list form and then in matrix form.

$x_1 + x_2$	$\geq 2$	1	1	$x_1$	>	2	
$-x_1 + 2x_2$	$\geq 0$	[-1]	2	$x_2$	_	0	

The H-description for Figure 2.1d is

$$\begin{array}{ccc} 2x_1 - x_2 &\geq 0 \\ -x_1 + 2x_2 &\geq 0 \end{array} \qquad \qquad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Notice that for the cone, the constant vector on the right is the zero vector. This is directly because every linear inequality for a cone must have the constant term b = 0(see Definition 2.1). Thus checking the constant vector of an *H*-description is a quick way to determine if a polyhedron is a cone.

If a polyhedron is a pointed cone, we have an additional way to describe it: as the nonnegative span of an irredundant set of vectors. This is called the *V*-description. A minimal description uses the extremal rays as the set of vectors. For Figure 2.1d, we can see that the rays (1, 2) and (2, 1) are the smallest integer-valued rays that fall along the hyperplanes of the halfspaces (in the image, these are the black bolded lines) in the direction of the polhedron. Although (-1, -2) and (-2, -1) also lie along these hyperplanes, a *V*-description takes only the *nonnegative* span, so using either or both of these will not span the correct area of the plane. So the *V*-description is

$$\operatorname{span}_{>0}\{(1,2),(2,1)\}.$$

**Theorem 2.3.** Let  $P \subseteq \mathbb{R}^n$  be an *n*-dimensional polyhedron. Then P has unique minimal H- and V-descriptions.

**Definition 2.4.** A *polytope* is a bounded polyhedron.

**Example 2.5.** Figure 2.2 shows a polytope formed from three halfspaces:  $2x_1 - x_2 \ge 0, -x_1 + 2x_2 \ge 0$ , and  $-x_2 \ge -2$ . The image on the left includes the



Figure 2.2. A polytope formed by the intersection of three halfspaces, shown with the bounding hyperplanes and without.

hyperplanes for better visualization of the three halfspaces, but the image on the right is just the polytope, the bounded interior of those hyperplanes. Note that the intersection of a finite number of halfspaces is not necessarily bounded, even if that finite number is large. These three hyperplanes divide the plane into 7 regions; any of these could be the intersection depending on which direction the inequality sign faces. Here, a single inequality sign change results in an unbounded polyhedron. We also note that, if all the inequality signs were reversed and we looked at the intersection of  $2x_1 - x_2 \leq 0, -x_1 + 2x_2 \leq 0$ , and  $-x_2 \leq -2$ , we would have the empty polytope since no ordered pairs satisfy all three inequalities at the same time.

Polytopes can also be described with an H-description, just like an unbounded polyhedron. For the polytope in Figure 2.2, the H-description is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

Polytopes do also have a V-description, but this takes a different form than that for cones. We first define the convex hull.

**Definition 2.6.** The *convex hull* of a set of points  $v_1, \ldots, v_k$  is the set of points satisfying a linear combination of  $v_1, \ldots, v_k$  with nonnegative coefficients and where the sum of the coefficients is 1. That is,

$$\operatorname{conv}\{v_1,\ldots,v_k\} = \{\lambda_1v_1 + \cdots + \lambda_kv_k \mid \lambda_i \ge 0, \ \lambda_1 + \cdots + \lambda_k = 1\}$$

While a cone's V-description is the nonnegative span of vectors, the polytope's V-description is the convex hull of a list of points. We can think of the convex hull like

	H-description	V-description (minimal)		
Cone	A list of halfspaces	The nonnegative span of vectors (rays)		
Polytope	A list of halfspaces	The convex hull of points (vertices)		
Table 2.1. The <i>H</i> - and <i>V</i> -descriptions for cones and polytopes.				



snapping a rubber band (or in higher dimensions, a balloon) around these points. A minimal description takes the set of vertices to be the points used. For our example polytope in Figure 2.2, the three pointed corners at the intersections of the hyperplanes are the vertices and so our V-description is

$$\operatorname{conv}\{(0,0), (1,2), (4,2)\}.$$

To summarize the H- and V-descriptions of cones and polytopes, we present Table 2.1. Taking a cross section of a cone can produce a polytope when done at the right angle. It is possible to take a cross section of a cone and yield an unbounded shape, but taking a cross section at a particular height (commonly, at height 1) or at a coordinate sum (commonly,  $x_1 + \cdots + x_n = 1$ ) typically yields a bounded polytope. Similarly, we can create a cone from a polytope by placing the polytope in a space one dimension higher at height 1 in that new dimension, and setting the extremal rays from the origin through the vertices of the polytope. In Figure 2.3, we take the polytope in (a) and create the cone in (b) through this process. We could conversely take the cone in (b), take a cross section at  $x_3 = 1$ , and obtain the polytope in (a).

**Definition 2.7.** For some polyhedron  $P, F \subseteq P$  is a *face* of P if there exists a half space containing P with boundary H such that  $F = H \cap P$ .



Figure 2.4. Two faces and a non-face of the unit cube.

**Example 2.8.** Consider the unit cube, Q, defined by the inequalities  $x_i \ge 0$  and  $x_i \le 1$  for i = 1, 2, 3. Note that when we use the matrix version of the *H*-description, inequality signs must all face the same direction, so instead of entering  $x_i \le 1$ , we enter  $-x_i \ge -1$ . We have

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ge \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$$
(2.2)

Consider the halfspaces

$$0x_1 + 0x_2 + 1x_3 \ge 0$$
,  $0x_1 + 0x_2 + 0x_3 \ge 0$ , and  $0x_1 + 0x_2 + 1x_3 \ge 0.5$ 

and their hyperplanes

$$H_1: x_3 = 0, \quad H_2: 0 = 0, \text{ and } H_3: x_3 = 0.5.$$

Figure 2.4 shows the intersection of the cube with each of these hyperplanes. Intersecting with  $H_1$  yields the bottom face in the xy plane (2.4a). Intersecting with  $H_2$ we have  $Q \cap H_2 = Q$ , so we see how a polyhedron is always a face of itself (2.4b). Intersecting with  $H_3$ , however, yields a slice through the middle of the cube. The halfspace with boundary  $H_3$ , regardless of whether  $\leq$  or  $\geq$ , does not contain Q. So the resulting intersection is not a face (2.4c). Figure 2.5 shows two more faces; one edge and one vertex. Faces range in dimension from 0 to the dimension of the entire polyhedron.

Note that the hyperplane H defining the face is not necessarily unique. In Figure 2.5a, the plane intersecting the edge could be at any angle between horizontal (where we would instead have the top face) and vertical (where we would instead have



(a) A 1-dimensional face.(b) A 0-dimensional face.Figure 2.5. Two more faces of the unit cube: an edge and a vertex.

the side face). Also, the hyperplane defining the face does not have to be from one of the halfspaces in the H-description. Thus it can be challenging, particularly in higher dimensions, to identify a single hyperplane to intersect with the polyhedron to define a particular face. Luckily, there is a more straightforward way.

**Definition 2.9.** Let P be an n-dimensional polyhedron. For faces of some specific dimensions we have specific names. We also use the word *codimension* to mean "dimensions less than the dimension of the polyhedron" and so we can refer to faces with either their dimension or their codimension.

dimension	name	$\operatorname{codimension}$
0	vertex	n
1	edge, ray	n-1
÷	:	÷
n-2	ridge	2
n-1	facet	1
n	polyhedron	0

Returning to the unit cube example, we have six 2-dimensional facets, twelve 1-dimensional ridges/edges, and eight 0-dimensional vertices. Including the polyhedron itself, we have a total of 27 faces. Each facet can be described by making one of the inequalities in the H-description in (2.2) into an equality. This results in all six facets, shown in Figure 2.6.

**Theorem 2.10.** The intersection of any two faces of a polyhedron yields another face. Further, every face can be written as an intersection of some collection of facets.

The above theorem tells us that, given an irredundant H-description for a polyhedron, we can identify its faces by taking subsets of the defining halfspaces and



Figure 2.6. The six facets of the unit cube, labeled with their facet equality.

making them equalities. Just like we did one-by-one with the unit cube to identify facets, we could take  $Q \cap (x_2 = 1) \cap (x_3 = 1)$  and have the edge highlighted in Figure 2.5a. The vertex in Figure 2.5b is  $Q \cap (x_2 = 1) \cap (x_3 = 1) \cap (x_1 = 1)$ . Notice that not every intersection of facets is unique. Both  $Q \cap (x_1 = 0) \cap (x_1 = 1)$  and  $Q \cap (x_3 = 0) \cap (x_3 = 1)$  result in the empty face, since the two planes represented by each pair of equalities are parallel. Nontrivial examples of nonunique intersections will be given in the next section.

The H-description of a face is taken to be the largest set of equalities (i.e., the largest collection of intersecting facets) which produce that face. This differs slightly from the definition of an H-description for a polyhedron in general, and is a more concise way to describe faces.

#### 2.2 The Kunz cone

In this thesis, we will study a particular family of cones, which will be defined by their *H*-description.

**Definition 2.11.** For  $m \ge 2$ , the Kunz cone  $C_m \subseteq \mathbb{R}_{\ge 0}^{m-1}$  is the set of all points  $(x_1, \ldots, x_{m-1})$  which satisfy

$$x_i + x_j \ge x_{i+j}$$
 for all  $1 \le i \le j \le m-1$ ,  $i+j \not\equiv 0$ 



where subscripts are interpreted modulo m. So if i + j > m, we take the subscript i + j - m. For  $m \ge 3$ , the set of inequalities forces  $x_i \ge 0$  for all  $i \in \{1, \ldots, m-1\}$ , so in these cases we need not specify  $\mathbb{R}^{m-1}_{\ge 0}$  and we will just write  $\mathbb{R}^{m-1}$ .

We will look at three examples:  $C_2$ , the smallest cone,  $C_3$ , the smallest interesting cone, and  $C_4$ , the next smallest. Since  $C_m \subseteq \mathbb{R}^{m-1}_{\geq 0}$ , these are the only three cones living in  $\mathbb{R}^3$  or smaller, and so they are the easiest to visualize.

**Example 2.12.** When m = 2, the list of inequalities is empty, so the cone is simply all of  $\mathbb{R}_{\geq 0}$ , shown in Figure 2.7a. The *H*-description is  $x_1 \geq 0$  and the *V*-description is  $\operatorname{span}_{\geq 0}\{(1)\}$ .

**Example 2.13.**  $C_3 \subseteq \mathbb{R}^2$  is defined by two inequalities, and has the *H*-description:

$$2x_1 \ge x_2$$
 and  $2x_2 \ge x_1$ .

There are two extremal rays, yielding the V-description  $\operatorname{span}_{\geq 0}\{(1,2),(2,1)\}$ . The cone is all of the points on and between these two rays, shown in Figure 2.7b. Note that this is also the cone shown in Figure 2.1d.

**Example 2.14.**  $C_4 \subseteq \mathbb{R}^3$  is defined by four inequalities, and has the *H*-description:

$$2x_1 \ge x_2$$
,  $x_1 + x_2 \ge x_3$ ,  $x_2 + x_3 \ge x_1$ , and  $2x_3 \ge x_2$ .

Each inequality forms a flat, 2-dimensional side of the cone, and four rays result from their intersections, yielding the V-description  $\operatorname{span}_{\geq 0}\{(1,0,1), (1,2,1), (1,2,3), (3,2,1)\}$ . We show  $C_4$  in Figure 2.7c.

In the context of the Kunz cone, we can think of a face as the resulting shape when some subset of the inequalities are made into equalities. For  $C_3$ , we have two



Figure 2.8. The faces of  $C_3$ .

inequalities and so we have four possible subsets:

Each of these subsets corresponds to a face of the cone, shown in Figure 2.8. The subset with no equalities gives the entire 2-dimensional cone, shown on the far left. In the middle we have the cases where we have one equality; these each produce a 1-dimensional ray and these are the two facets of  $C_3$ . On the right we have the case where both are equalities; the face is restricted to the vertex at (0,0).

We now look to  $C_4$  and its faces, shown in Figure 2.9. We have the entire 3-dimensional cone when no inequalities are equalities (bottom row, left), and the 0-dimensional vertex when all are equalities (bottom row, right). In between dimensions 3 and 0, we have the four flat 2-dimensional facets from when we have just one equality (top row), and we have four 1-dimensional rays/ridges (middle row), located where two facets intersect and thus each of these faces have two inequalities and two equalities. However, notice that with four inequalities, there are  $2^4 = 16$  subsets of inequalities to make equalities, and yet we have only ten faces listed. This is because, as was mentioned in the context of the unit cube, not every subset describes a unique face. The two-dimensional faces on opposite sides of the cone intersect in only the vertex. Further, any subset of three equalities also restricts to only the vertex. So there are 7 subsets which give only the vertex. Since we define a face with the largest set of intersecting facets, as mentioned in Theorem 2.10, we present it with every inequality an equality. It is not always easy, particularly in higher dimensions, to determine the dimension of a face just from picking an arbitrary set of facets to intersect.

It is well understood that since a single equality results in a facet, the number of facets is equal to the number of inequalities which describe the cone (a formula will be given in Chapter 4). What is less well understood, but we hope to describe in Chapter 4, is how to determine when the intersection of facets results in a ridge versus when it

results in a lower-dimensional face, and by extension, how many ridges a given Kunz cone has.

#### 2.3 Posets

As previously mentioned, we can use the *H*-description of a face to identify it. Because a list of inequalities can quickly get lengthy, though, we introduce the following as a neater way to uniquely identify most faces of  $C_m$ .

**Definition 2.15.** A *poset* is a partially ordered set, using the order  $\leq$ . Let F be a face of the Kunz cone  $C_m$  in which there exists a point  $(x_1, \ldots, x_{m-1})$  with  $x_i > 0$  for all  $i \in \mathbb{Z}_m \setminus \{0\}$ . To construct a poset for F, we use the set  $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$  and we have the following properties on  $\leq$ :

- $0 \leq i$ , for all  $i \in \mathbb{Z}_m$ ;
- $i \leq k \iff x_i + x_j = x_k$  is in the *H*-description of the face.

The ordering  $\leq$  is reflexive, transitive, and antisymmetric. We represent the poset with a Hasse diagram, drawn with the following rule: When  $a \leq b$ , we write a underneath b in the diagram. If  $a \leq b$  and there does not exist any c such that  $a \leq c \leq b$ , we draw a line from a to b. We say a is a maximal element if for all  $b \in \mathbb{Z}_m$ ,  $a \not\leq b$ .

**Example 2.16.** Let m = 4 and consider the case when the last two inequalities of (2.14) are equalities. We have  $x_2 + x_3 = x_1$  and  $2x_3 = x_2$ . Thus  $2 \leq 1, 3 \leq 1$ , and  $3 \leq 2$ . We have the following poset:

$$\begin{array}{c}\bullet 1\\\bullet 2\\\bullet 3\\\bullet 0\end{array}$$

Notice we have each of these relations present in the diagram. Both 2 and 3 are below and connected to 1, and 3 is below and connected to 2. In this poset, 1 is the only maximal element. Notice that we have only the elements  $\{0, 1, 2, 3\}$  in this set, and for every pair of elements a, b, either  $a \leq b$  or  $b \leq a$ . This results in a poset that is a vertical line and we call this a *totally ordered* poset.

**Example 2.17.** Figure 2.10 shows three posets representing faces of  $C_9$ . The center and right posets both have a single maximal element: 2. The left poset, though, has 5 maximal elements: 4, 6, 5, 7, and 8. This example is included to show how posets can quickly get complex when m gets large and when the number of relations increases.



Figure 2.9. The faces of  $C_4$ .

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Figure 2.10. Three posets representing faces in  $C_9$ .



Figure 2.11. Posets for each face of  $C_3$ .

Each face of a cone has a unique poset, so posets can be a useful way to represent faces. We look at the posets for each nontrivial face of  $C_3$ . Figure 2.11 shows each face with the equalities that define it, an image of what part of the cone it is, and the unique poset that represents it. Figure 2.12 shows a few faces of  $C_4$  with their respective equalities and posets.

Consider the ray (1, 0, 1) defined by the intersection of  $F_1 : (x_1 + x_2 = x_3)$  and  $F_2 : (x_2 + x_3 = x_1)$ . For any point along this ray,  $x_2 = 0$ , so Definition 2.15 does not apply. Were we to try to use that definition, we would find that both  $1 \leq 3$  and  $3 \leq 1$ . This contradiction results in a collapsed poset where 1 and 3 occupy the same vertex and 0 and 2 occupy the same vertex. Note that 2 divides 4 and  $x_2$  is the coordinate equal to zero. In general, when  $2 \mid m$ , there exists a ray of  $C_m$  which takes the form of  $x_i = 1$  when  $i \equiv 1 \mod 2$  and  $x_i = 0$  when  $i \equiv 0 \mod 2$ . For example, there exists a ray (1, 0, 1, 0, 1) in  $C_6$  and a ray (1, 0, 1, 0, 1) in  $C_8$ . More generally, when  $d \mid m$ , there exists an injective map  $C_d \hookrightarrow C_m$  in which faces in  $C_d$  map to faces in  $C_m$  (see [4] for a more thorough exploration). Necessarily, these faces are of the same dimension in  $C_m$  as



Figure 2.12. The posets of four faces of  $C_4$ .



Figure 2.13. Two posets of  $C_4$  exhibiting collapse. On the left, the poset for the ray (1,0,1), and on the right, the poset for the vertex.

they are in  $C_d$ , and their posets exhibit the exact same structure as they do in  $C_d$ , with elements of  $\mathbb{Z}_m$  in the same equivalence class modulo d sharing the same vertex. The poset for the vertex of any  $C_m$  has a similar issue, and the poset collapses to a single vertex. See Figure 2.13 for the poset of the ray (1,0,1) and the vertex (0,0,0) of  $C_4$ .

#### 2.4 Projections

To visualize shapes in dimensions higher than 3, we need ways to project those shapes down into fewer dimensions without losing important information. There are two ways we will do this: taking a cross section and making a Schlegel diagram.

Taking a cross section of a cone in the right way gives a bounded, lower-dimensional shape. In this project, we take a cross section at a coordinate sum of 1. That is, we define the projection as the shape  $\hat{C}_m$  so that

$$\hat{C}_m = C_m \cap \{x_1 + x_2 + \dots + x_{m-1} = 1\}.$$

 $\hat{C}_3$  gives a line segment, and  $\hat{C}_4$  gives a kite, as shown in Figure 2.14. Notice that each ray becomes a vertex, each 2-dimensional face becomes a 1-dimensional edge, and the 3-dimensional interior of the cone is represented by the 2-dimensional interior of the cross section. The cross section lives in one fewer dimension, but we preserve the face structure: for example, any two facets of  $C_4$  which intersect in a ray are shown as edges



Figure 2.14.  $\hat{C}_4$  is a kite.



Figure 2.15. Two projections of a cube from 3 dimensions into 2 using a Schlegel diagram, done via popping two different faces.

intersecting in a vertex in  $\hat{C}_4$ . The cross section drawn here is proportional to the actual measurements. The bottom angle is 60°; in fact, the bottom triangle (drawing a horizontal line between the left and right vertices) gives an equilateral triangle. The remaining angles are approximately (but not exactly) 100°. It is possible, with the intersecting plane  $x_1 + x_3 = 4$ , to obtain a rhombus cross section with equal side lengths, but the diagonals are different lengths and so a square is not possible.

We can also use a Schlegel diagram to project the cone into one fewer dimension by puncturing a face. We can think of this like "popping" one face open and folding the shape "flat" in the lower dimension with the "popped" face as the outside face. We show this done on the cube Q to create a 2-dimensional projection in Figure 2.15. We present two different Schlegel diagrams, created by puncturing a different face. Figure 2.15a uses a facet: the shaded facet, closest to us in the 3-dimensional image on the left, becomes the outside face in the 2-dimensional image on the right, while every other facet has been folded out to lie flat. Notice that we still have all the information about relationships between the faces. Each vertex still has degree 3, each edge still connects the same vertices and adjacent facets are still adjacent. Angles and lengths have been skewed, but because we are able to preserve this important connectivity information, these projections are valuable techniques to help us visualize shapes in higher dimensions. Figure 2.15b uses a ridge: the highlighted edge on the left of the cube is sent to infinity up and down, splitting the page into a left and a right. There are now two outside facets. Together with the four still bounded facets, we can still identify the six facets, and they all still have four edges (recall that the upper and lower highlighted lines are from the punctured edge and so count as the same edge). All the same information is preserved.

In Chapter 3, these two projection techniques will be used to create 3-dimensional diagrams for the Kunz cones which live in in 4-, 5-, and 6-dimensions.

#### CHAPTER 3

#### Examination of $C_5$ , $C_6$ , and $C_7$

Having already examined  $C_2$ ,  $C_3$ , and  $C_4$ , we turn our attention to the next three Kunz cones in an attempt to understand their face structures.

#### **3.1** The Kunz cone $C_5$

We first look at  $C_5 \subseteq \mathbb{R}^4$ , since it is the first Kunz cone larger than 3 dimensions and thus the simplest non-visualizable cone. A table of all the faces of  $C_5$  with their dimensions is presented in Table 3.1. We also include the breakdown of the 3-dimensional facets into how many rays they involve. Four of the eight facets contain 3 rays and the other four contain 4 rays. However, this table does not show us any relationships between faces of different dimensions.

We introduce a diagram called a face lattice which organizes the face structure of the Kunz cone. We use posets to identify each face in the face lattice, arrange them by dimension, and use connecting lines to show containment. Figure 3.1 is the full face lattice for  $C_5$ . The uppermost row has the poset representing the 4-dimensional interior of the cone. Below that in the second row are the eight 3-dimensional facets, below those in the third row are the fourteen 2-dimensional ridges, and the bottom row contains the eight 1-dimensional rays. We can see that, for example, the totally ordered poset in the bottom left of the face lattice represents a ray that is contained in four different ridges. Similarly, the poset on the leftmost side of the third row represents a ridge which contains two rays and is contained in two facets.

The edges that are bolded highlight connections between posets which have 2 as a maximal element, and Figure 3.2 shows a sublattice using only these edges and posets. In this smaller, more focused lattice, it is easier to trace poset relations. For example, we can look for all the posets in which  $4 \leq 3$ . There is one facet that does so; it is located in the middle of the second row. Working down the lattice, we can follow the containment lines and see there are two ridges that it contains: the third and sixth on the third row. Finally, there is just one ray in which  $4 \leq 3$ : the one on the right of the bottom row. These are all of the faces which involve the equality  $2x_4 = x_3$  and have 2 as a maximal element.

Face dimension	Number of faces		
0	1 vertex		
1	8 rays		
2	14 ridges		
3	8 facets	4 3-ray	
	0 140005	4 4-ray	
4	1 polyhedron		
Total	32 faces		

Table 3.1. The faces of  $C_5$ , by dimension.

Taking a cross section at the coordinate sum of 1 ( $\hat{C}_5$ ), we obtain a shape called an irregular gyrobifastigium<sup>1</sup>. We can assign posets to the diagram of this projection to further understand the shape and connectivity of  $C_5$ . Figure 3.4a shows this cross section with the ray posets next to their corresponding vertices, Figure 3.4b shows it with the ridge posets next to their corresponding edges, and Figure 3.4c shows it with the facet posets and their corresponding sides. Notice that posets of matching shapes correspond to matching shapes within the cross section. For example, the fully ordered ray poset family corresponds to the four rays along the center of the gyrobifastigium while the ray posets with the more asymmetrical shape correspond to the rays on the outsides of the gyrobifastigium. We can also see that the posets corresponding to quadrilateral facets take one shape while the posets corresponding to triangular facets take another.

Figure 3.3 presents the actual proportions of the faces of  $\hat{C}_5$ . We see that we do not have a regular gyrobifastigium. The triangular faces are isosceles triangles with approximate interior angles of 48° and 84° and side lengths of  $10\sqrt{5}$  and 30. The quadrilateral faces are isosceles trapezoids with approximate interior angles of 77° and  $103^{\circ}$  and side lengths of 30 and 20 on the bases and  $10\sqrt{5}$  on the lateral sides. It may be possible to obtain different proportions with respect to angles and side lengths if the cross section is taken at a different angle (not at a fixed coordinate sum), but this is a topic for a future research project.

<sup>&</sup>lt;sup>1</sup>The gyrobifastigium is the 26th Johnson solid. It consists of two triangular prisms attached along a square side. Technically, the gyrobifastigium is face-regular; this is not necessarily guaranteed with  $\hat{C}_5$  or any later time this shape appears in this text. Specific coordinates, angles, and lengths of the shapes we will see in this text are chosen to highlight symmetry, ensure coplanarity when needed, and produce a shape that is as easy as possible to look at and learn relevant information. Faces of the Kunz cone have not been studied enough to determine regularity, but for lack of an existing word to describe an irregular gyrobifastigium, we will call it a gyrobifastigium.







Figure 3.2. A sublattice of the full face lattice of  $C_5$  using only posets in which 2 is a maximal element.



(a) The trapezoidal faces (b) The triangular faces

Figure 3.3. The proportions of the gyrobifastigium in  $\hat{C}_5$  for a fixed coordinate sum.





Face dimension	Number of faces		
0	1 vertex		
1	11 rays		
2	29		
2	30 ridros	22 3-ray	
0	50 Huges	8 4-ray	
		4 4-ray	
4	19 facots	4 5-ray	
±	12 lacets	2 6-ray	
		2 7-ray	
5	1 polyhedron		
Total	84 faces		

Table 3.2. The faces of  $C_6$ , by dimension.

#### **3.2** The Kunz cone $C_6$

We now turn to the 5-dimensional  $C_6$ . A table of the faces of  $C_6$  and their dimensions is shown in Table 3.2.

A face lattice would be quite large at this point, with 84 total faces. What we aim to accomplish instead is creating a diagram like that in Figure 3.4. so we can understand how the different faces relate to each other – which vertices have an edge between them, which edges outline the faces of ridges, and which ridges form 3-dimensional chambers. Because we now have two dimensions to reduce by projections, we first take a cross section at the coordinate sum of 1 ( $\hat{C}_6$ ), and then puncture a face to obtain a Schlegel diagram.

Figure 3.5a shows the vertex graph produced by SageMath: each ray is represented with a vertex, and each 2-dimensional face spanned by two rays is represented with an edge between those vertices, but this is the only information the graph contains. This graph is a bit messy with the labeling of the vertices and it has no depth so determining any 3-dimensional chambers is nigh impossible. Even 2-dimensional faces are difficult to determine. SageMath can, however, produce a Schlegel diagram, which does include some depth perspective and is a 3-dimensional object instead of a flat 2-dimensional graph. Figure 3.5b shows the Schlegel diagram that SageMath produces for  $\hat{C}_6$ . While chambers are now distinct, there is no apparent symmetry. This is something we hope to fix by manually choosing the location of each vertex.

In constructing a Schlegel diagram, we are not concerned with accuracy in terms of edge lengths and angles, but we aim to preserve any symmetry and also to ensure



-0.07

(b) The Schlegel projection of  $\hat{C}_6$ .

v=0.13

x=0.14

Figure 3.5. The vertex graph and Schlegel projection of  $\hat{C}_6$  automatically created by SageMath.

0.25

Graph on 11 vertices

46.0, 1/3, 1/6

ertex at ()

tex at (1/15 2)15

at (1/92)9, 1/3, 2/9, 1/9)

H 3/8, 1/3, 0/ 1/3)

A vertex at (1/9 2)9, 1/3, 1/9, 2/9)

(a) The vertex graph of  $\hat{C}_6$ .

A vertex at (1/7 27, 1/7, 2/7, 1/7)

A vertex at (1/3,

A vertex

A vertex at (1/6, 1/2, 1/4, 1/6, 1/1

vertex at (1/1

1/5. 2/1 1/

coplanarity of 2-dimensional faces. Each 3-dimensional chamber within the diagram must not overlap with any others and should have enough volume to be identifiable as 3-dimensional and not flat.

We found that  $C_6$  has 180° rotational symmetry along an axis of three vertices. Due to this symmetry, the punctured face is actually a 3-dimensional ridge, not a facet. In the diagram, it is a 2-dimensional face. See Figures 3.6 and 3.7 for the completed 3-dimensional projection of  $C_6$ . In Figure 3.6 we use darker colors to show faces closer to the reader and lighter colors to show faces further away. View 1 in 3.6a shows the diagram at an angle in which every vertex and edge is visible and non-overlapping. Figure 3.7 uses this same angle, but highlights important features of the diagram. In 3.7a, we highlight the three vertices on the axis of symmetry. Rotation by  $180^{\circ}$  around this axis yields an indistinguishable view. View 2 in 3.6b shows the diagram when looking directly down the axis of symmetry. The symmetry is perhaps more apparent here, where we can see pairs of vertices which rotate to each other's position: top and bottom; upper right and lower left; upper left and lower right; and the two internal vertices appearing just above and below that axis.

There are twelve 3-dimensional chambers in this diagram. Due to symmetry, these chambers come in six symmetric pairs. Five of these pairs (excluding the outside facets) are shown in Appendix A using the same viewpoints as in 3.6.

In Figure 3.7b, we highlight the four edges at the boundary of the punctured ridge, and in 3.7c we highlight the exterior faces. The external shape in this projection again appears as a gyrobifastigium – the same square which is the border of the punctured ridge marks the border of the square base where two triangular prisms are



Figure 3.6. Two views of the Schlegel diagram of  $\hat{C}_6$  in 3 dimensions.



(a) The axis of symmetry (b) The punctured ridge (c) All exterior faces Figure 3.7. The Schlegel diagram of  $\hat{C}_6$  with some key features highlighted.

glued. We remind readers that since angles and proportionality of edge lengths are not preserved, we cannot actually claim that the shape is a regular gyrobifastigium, that any quadrilateral faces are squares, or that any triangles are equilateral. However, these projections are still useful and applying some level of face-regularity makes decoding the embedded information easier. Further, in  $C_6$ , the punctured ridge means that the gyrobifastigium we see in the Schlegel diagram is actually the outline of two facets split by a ridge, not one single face. However, the reappearance of this shape in any context is worth noting here and should be considered for pursuing in future research.

Face dimension	Number of faces		
0	1 vertex		
1	30 rays		
2	114		
2	159	104 3-ray	
0	102	48 4-ray	
		15 4-ray	
		24 5-ray	
4	84 ridges	24 6-ray	
		12 7-ray	
		9 8-ray	
		6 9-ray	
5	18 facets	6 11-ray	
		6 14-ray	
6	1 polyhedron		
Total	400 faces		

Table 3.3. The faces of  $C_7$ , by dimension.

#### **3.3** The Kunz cone $C_7$

We now turn to  $C_7 \subseteq \mathbb{R}^6$ . A table of the faces of  $C_7$  and their dimensions is shown in Table 3.3.

Since we currently only have the two projection techniques explained in 2.4, they only allow us to bring  $C_7$  down to 4 dimensions, which is still not particularly visualizable. Instead, we apply these projections to each facet of  $C_7$ , which are 5-dimensional, and so, after two projections, are represented in 3 dimensions. These facets do not have internal symmetry, but we do still have the challenges of coplanarity for 4-vertex faces and ensuring no 3-dimensional chambers have 3-dimensional intersections. There are 18 facets of  $C_7$ , but they come in three groups of six. Six facets are made with 9 rays, six made with 11 rays, and six made with 14 rays. Within these groups, they are the same up to symmetry, so it suffices to focus on just one from each group.

For the 9-ray facet, we present Figure 3.8a. The outside face has 7 vertices and was chosen to be the outside face simply because it had the largest number of vertices. They have been placed at corners of the unit cube, minus one, and the remaining two vertices are inside. Four 2-dimensional, 4-vertex faces needed to be kept coplanar; three of these are external on sides of the "cube" while the fourth involves the two internal vertices and the two vertices in the top back. The remaining seventeen 2-dimensional faces involve just 3 vertices. Of the 3-dimensional chambers, three involve just 4

vertices, two involve 5 vertices, two involve 6 vertices, and as mentioned above, one (the outside face) involves all 7.

The 11-ray facet is given in Figure 3.8b. Like  $C_6$  and  $C_5$ , the external shape appears as a gyrobifastigium. This is partly choice, as that face was chosen to be the external one on the basis of it having the most vertices for a single chamber, and so it is more visible here, yet it still is surprising that this shape appears again. There are eight 2-dimensional, 4-vertex faces needing to be kept coplanar; four of these are external and part of the outside gyrobifastigium face, and four are internal. The remaining seventeen 2-dimensional faces involve just 3 vertices. Of the 3-dimensional chambers, two involve 4 vertices, two involve 5 vertices, three involve 6 vertices, one involves 7, and one involves 8.

The 14-ray facet is given in Figure 3.8c. Again, the gyrobifastigium shape appears. Here, there were two chambers involving 8 vertices: one took the gyrobifastigium shape, and the other took a shape with an identical face lattice to a cube (six 4-edge sides). The gyrobifastigium was chosen to be the outside face to highlight the frequency with which this shape appears in the Kunz cone. In addition to the two 8-vertex chambers, there are four 5-vertex, three 6-vertex, and two 7-vertex chambers. There are thirteen 4-vertex 2-dimensional faces, and the remaining twenty 2-dimensional faces involve just 3 vertices.

What has not yet been determined is how each of these facet types intersect with each other. Perhaps there is a different set of Schlegel diagrams which more easily highlight how the facets intersect within  $C_7$ .

#### **3.4** Summary of $C_5$ , $C_6$ , and $C_7$

When we look at the face structures of  $C_5$ ,  $C_6$ , and the facets of  $C_7$ , some patterns arise. A notable one is the recurrence of the irregular gyrobifastigium shape which was present in  $C_5$ ,  $C_6$ , and two of the facets of  $C_7$ . Future research could be done on the prevalence of this shape occurring in Kunz cones generally and determining what factors might predict it or if there are certain families of faces in which it occurs. Another pattern that was observed is that all 3-dimensional faces so far involve only 3 or 4 rays. None were found that involve more than 4. Whether this continues in higher dimensions still could be another area of future research<sup>2</sup>. In the cones studied here, no 4-dimensional faces involved more than 8 rays, but 9-ray 4-dimensional faces can be found in  $C_9$  and  $C_{10}$ . It would be interesting to determine if there is a limit on the number of rays involved in 4-dimensional faces.

<sup>&</sup>lt;sup>2</sup>Work done by Joe McDonough and Cole Brower, communicated via private correspondence, suggests a bound at 4 exists for the number of rays in 3-dimensional faces, but it has yet to be proved.



Figure 3.8. The Schlegel diagram for each facet type of  $C_7$ .

## CHAPTER 4 Ridges

Every ridge is contained in exactly two facets, but not every pair of facets intersect in a ridge. Recall the example in Chapter 2 with  $C_4$ . Adjacent facets intersect in a ridge, but facets opposite each other intersect in the vertex. In this chapter, we aim to describe when two facets intersect in a ridge versus when they intersect in a lower-dimensional face. We use

$$F_1: x_i + x_j = x_{i+j}$$
 and  $F_2: x_k + x_l = x_{k+l}$ 

to refer to the two facets we are intersecting. We begin by defining three different types of faces and then conclude when, given two arbitrary facets, their intersection is codimension 2 or codimension > 2.

**Definition 4.1.** Let m = 4n, for some  $n \in \mathbb{Z}$ ,  $n \ge 2$ . For  $i \in \{1, \ldots, n-1\}$ , set j = i + 2n, k = i + n, and l = i + 3n. We form the equations

$$F_1: x_i + x_j = x_{i+j}$$
$$F_2: x_k + x_l = x_{k+l}.$$

The face of  $C_m$  defined by  $F_1 \cap F_2$  is called a quadribigeminal<sup>1</sup> face

**Lemma 4.2.** Quadribigeminal faces have codimension > 2 and thus are not ridges. Moreover, the number of quadribigeminal faces in  $C_m$  is n-1 if m = 4n and 0 otherwise. That is,

$$\delta_q = \begin{cases} \frac{1}{4}m - 1 & \text{if } m \equiv 0 \mod 4\\ 0 & \text{if } m \not\equiv 0 \mod 4 \end{cases}.$$
(4.1)

*Proof.* First, we note that quadribigeminal faces are defined only when  $4 \mid m$ , so there are 0 of them when  $4 \nmid m$ . When m = 4n, the subscripts i, j, k, l belong to the same nonzero equivalence class in  $\mathbb{Z}_m/\langle n \rangle$  so to count the number of quadribigeminal faces it suffices to count the number of nonzero equivalence classes, which is n - 1.

<sup>&</sup>lt;sup>1</sup>From the Latin "quad" meaning "four", "bi" meaning "two", and "geminus" meaning "twin". We have four unique subscripts i, j, k, l which, when doubled, produce two unique sums i + j and k + j. In this way, i and j are "twins" since their doubles are the same, and k and l are "twins" since their doubles are the same.

Now, notice that

$$i + j = 2i + 2n = 2i + \frac{1}{2}m \equiv 2l \equiv 2k$$
  
 $k + l = 2i + 4n = 2i + m \equiv 2i \equiv 2j.$ 

This means we can equally write

$$F_1 \text{ as } x_i + x_j = x_{2k} \text{ or } x_i + x_j = x_{2l}, \tag{4.2}$$

and similarly, we could write

$$F_2 \text{ as } x_k + x_l = x_{2i} \text{ or } x_k + x_l = x_{2j}.$$
(4.3)

From the inequalities forming  $C_m$ , we have

$$2x_i \ge x_{2i}, \quad 2x_j \ge x_{2j}, \quad 2x_k \ge x_{2k}, \quad \text{and} \quad 2x_l \ge x_{2l},$$

$$(4.4)$$

and so, adding these together, we have

$$2x_i + 2x_j + 2x_k + 2x_l \ge x_{2i} + x_{2j} + x_{2k} + x_{2l}.$$
(4.5)

Adding  $2F_1$  and  $2F_2$ , and using (4.2) and (4.3), we have

$$2x_i + 2x_j + 2x_k + 2x_l = x_{2l} + x_{2k} + x_{2i} + x_{2j}.$$
(4.6)

Combining (4.5) and (4.6) (the latter in reversed order), we have

$$2x_i + 2x_j + 2x_k + 2x_l \ge x_{2i} + x_{2j} + x_{2k} + x_{2l} = 2x_i + 2x_j + 2x_k + 2x_l.$$

$$(4.7)$$

Since the left and right hand sides are identical, we in fact have equality and not inequality in (4.5) and (4.4). Thus the face defined by  $F_1$  and  $F_2$  has the additional equalities

$$2x_i = x_{2i}, \quad 2x_j = x_{2j}, \quad 2x_k = x_{2k}, \quad \text{and} \quad 2x_l = x_{2l}$$

Because a ridge is contained in exactly two facets, and these quadribigeminal faces are contained in at least six, we can conclude that these faces are not ridges.  $\Box$ 

**Example 4.3.** Consider m = 8, the smallest Kunz cone to have a quadribigeminal face. We have m = 8 = 4(2), so n = 2. So there is only one quadribigeminal face, the intersection of

$$F_1: x_1 + x_5 = x_6$$
 and  $F_2: x_3 + x_7 = x_2$ .

Figure 4.1 shows a poset for this face. From  $F_1$  and  $F_2$  we have  $1 \leq 6, 5 \leq 6, 3 \leq 2$ , and  $7 \leq 2$ . However, the additional equalities given in 4.7 are

$$2x_1 = x_2$$
,  $2x_5 = x_2$ ,  $2x_3 = x_6$ , and  $2x_7 = x_6$ 

so the poset has the additional relations  $1 \leq 2, 5 \leq 2, 3 \leq 6$ , and  $7 \leq 6$ .



Figure 4.1. A poset for the quadribigeminal face of  $C_8$ .

Recall the collapse that occurred in  $C_4$  on the ray (1, 0, 1) (See Figure 2.13). As mentioned previously, when  $d \mid m$  there exists an injective dimension-preserving map in which faces in  $C_d$  map to faces in  $C_m$ . As a divisor of  $m, d \leq \frac{m}{2}$ , so  $C_d$  has dimension at most  $d - 1 \leq \frac{m}{2} - 1$ . For a face  $F \subseteq C_m$  to be a ridge, it must be that F has dimension m - 3 (codimension 2 in an m - 1 dimensional object). That means if a ridge is a collapsed face,  $m - 3 \leq \frac{m}{2} - 1$  and, solving for m, we have  $m \leq 4$ . To avoid collapsed ridges, we give the following theorems for  $m \geq 5$ .

**Definition 4.4.** Let  $i, j, k, l \in \mathbb{Z}_m \setminus \{0\}$  such that

$$i+j \not\equiv k, \quad i+j \not\equiv l, \quad k+l \not\equiv i, \quad \text{and} \quad k+l \not\equiv j.$$

We set

$$F_1: x_i + x_j = x_{i+j}$$
$$F_2: x_k + x_l = x_{k+l}.$$

Supposing  $F_1 \neq F_2$ , the face defined by  $F_1 \cap F_2$  is called an *esker*<sup>2</sup> face.

**Lemma 4.5.** For  $m \ge 5$ , an esker face is a ridge if and only if it is not a quadribigeminal face.

*Proof.* We have already shown that quadribigeminal faces are not ridges; it remains to show that esker faces which are not quadribigeminal are ridges. So we suppose that  $F_1 \cap F_2$  is not a quadribigeminal face. We construct the point

<sup>&</sup>lt;sup>2</sup>Derived from the Irish word "eiscir," an esker is a geological ridge of stratified sand and gravel.

and  $x_e = 7m$ , for  $e \in \{1, ..., m-1\} \setminus \{i, j, k, l, i+j, k+l\}$ . It is clear that  $x_i + x_j = x_{i+j}$  and  $x_k + x_l = x_{k+l}$  are satisfied. For the remaining inequalities, we will divide the  $x_i$  into three sets:

$$S_{1} = \{x_{i}, x_{j}, x_{k}, x_{l}\}$$
$$S_{2} = \{x_{i+j}, x_{k+l}\}$$
$$S_{3} = \{x_{e} \mid x_{e} \notin \{S_{1} \cup S_{2}\}\}$$

Note that 6m is an upper bound on  $S_1$ , 12m is an upper bound on  $S_2$ , and 7m is an upper bound on  $S_3$  (in fact, every element in  $S_3$  is 7m). We have two cases for the remaining Kunz inequalities  $y_a + y_b \ge y_{a+b}$ . First, suppose  $y_a, y_b \in S_1$ . Note that we have already addressed the case of summing  $x_i$  and  $x_j$  and the case of  $x_k$  and  $x_l$ , so suppose  $y_a$  and  $y_b$  are some other pair. Because  $F_1 \cap F_2$  is not a quadribigeminal face, no other pair adds to any element within  $S_2$ , so  $y_{a+b} \in S_1 \cup S_3$  and so  $y_{a+b} \le 7m$ . Since  $y_a, y_b > 5m$ ,  $y_a + y_b > 10m > 7m \ge y_{a+b}$ .

Next, we choose  $y_a$  and  $y_b$  so that they are not both in  $S_1$ . Note that  $y_{a+b}$  could be in any set, so  $y_{a+b} < 12m$ . We have that  $y_a + y_b > 12m > y_{a+b}$ .

In both cases, the Kunz inequality  $y_a + y_b \ge y_{a+b}$  is satisfied with strict inequality, so we have a ridge.

Going forward, we will use the term "esker ridge" to refer to a face which is both an esker face and a ridge (i.e., not a quadribigeminal face). Finally, we describe a third type of face.

**Definition 4.6.** Let  $i \in \{1, \ldots, m-1\} \setminus \{\frac{m}{3}, \frac{2m}{3}, \frac{m}{2}\}$  and let j = k = i and l = 2i so that

$$F_1 : 2x_i = x_{2i}$$
$$F_2 : x_i + x_{2i} = x_{3i}.$$

The face defined by  $F_1 \cap F_2$  is called a *tripling face*.

Lemma 4.7. A tripling face is codimension 2 and thus a ridge.

*Proof.* We construct the point

$$x_i = 5m + i$$
,  $x_{2i} = 10m + 2i$ ,  $x_{3i} = 15m + 3i$ , and  $x_e = 8m_e$ 

$$\begin{aligned} x_i + x_{3i} &= 20m + 4i & x_{2i} + x_e &= 18m + 2i \\ x_i + x_e &= 13m + i & x_{3i} + x_{3i} &= 30m + 6i \\ x_{2i} + x_{2i} &= 20m + 4i & x_{3i} + x_e &= 23m + 3i \\ x_{2i} + x_{3i} &= 25m + 5i & x_e + x_e &= 16m. \end{aligned}$$

The only value of the above sums that is less than any individual entry is  $x_i + x_e = 13m + 1$ . Since  $e \neq 2i$ ,  $x_i + x_e \neq x_{3i}$ , and the sum 13m + 1 is larger than every other individual entry. So the given point satisfies every remaining Kunz inequality with strict inequality, and so we have a ridge.

Corollary 4.8. The number of tripling ridges is given by

$$\delta_t = \begin{cases} m - 4 & \text{if } m \equiv 0 \mod 6 \\ m - 1 & \text{if } m \equiv 1 \mod 6 \\ m - 2 & \text{if } m \equiv 2 \mod 6 \\ m - 3 & \text{if } m \equiv 3 \mod 6 \\ m - 2 & \text{if } m \equiv 4 \mod 6 \\ m - 1 & \text{if } m \equiv 5 \mod 6 \end{cases}$$
(4.8)

*Proof.* In any Kunz inequality, subscripts must be nonzero. By picking  $i \in \{1, \ldots, m-1\}$  we ensure i is nonzero, but we must also ensure  $2i \not\equiv 0 \mod m$  and  $3i \not\equiv 0 \mod m$ . Thus  $i \notin \{\frac{m}{2}, \frac{m}{3}, \frac{2m}{3}\}$ . If  $2 \nmid m$ , then  $\frac{m}{2} \notin \mathbb{Z}$ . If  $3 \nmid m$ , then  $\frac{m}{3} \notin \mathbb{Z}$  and  $\frac{2m}{3} \notin \mathbb{Z}$ . So the number of tripling ridges is m-1-c where c is how many of  $\{\frac{m}{2}, \frac{m}{3}, \frac{2m}{3}\}$  are integers. Checking cases for m modulo 6, this yields the above formula.

When trying to find a rule for when two facets intersect in a ridge, it initially seemed that declaring that  $i + j \neq k, l$  and  $k + l \neq i, j$  was sufficient. However, it became clear after multiple tests that there existed some ridges that this rule did not count as ridges and that there existed some nonridges that this rule did count as ridges. These two cases led to the definitions of the quadribigeminal and the tripling faces. The quadribigeminal faces were counted as ridges but weren't, and the tripling faces were not counted as ridges but are. This leads us to the following theorem.

**Theorem 4.9.** Let  $m \ge 5$ . Every ridge of  $C_m$  is either a tripling ridge or an esker ridge.

Proof. Let  $F_1: x_i + x_j = x_{i+j}$  and  $F_2: x_k + x_l = x_{k+l}$  such that  $F_1 \neq F_2$ . If  $F_1 \cap F_2$  is an esker ridge, we are done, so suppose not. We will show that to be a ridge it must be a tripling ridge. Without loss of generality, suppose i + j = k. From  $F_1$  we have the poset relation  $i \leq i+j$  and from  $F_2$  we have the poset relation  $i + j \leq i+j+l$ . By transitivity of  $\leq$ , we must have  $i \leq i+j+l$ , meaning there must exist an equation  $x_i + y = x_{i+j+l}^3$ . By construction of the Kunz cone,  $y = x_{j+l}$ . Similarly, we also have  $j \leq i+j+l$  and so there must exist  $x_j + x_{i+l} = x_{i+j+l}$  to define this face. Since  $F_1 \cap F_2$  is a ridge, neither  $G_1: x_i + x_{j+l} = x_{i+j+l}$  nor  $G_2: x_j + x_{i+l} = x_{i+j+l}$  are distinct equations from  $F_1$  and  $F_2$ .

- 1. Suppose  $G_1 = F_1$ . Since both equations involve  $x_i$  on the left hand side, we must have  $x_j = x_{j+l}$  and thus l = 0. But l cannot be zero since all subscripts must be nonzero so  $G_1 \neq F_1$ .
- 2. Suppose  $G_1 = F_2$ . We must be adding the same two numbers together so either  $x_i = x_{i+j}$  or  $x_i = x_l$ . If  $x_i = x_{i+j}$  then j = 0 and this is a contradiction. So i = l.
- 3. Suppose  $G_2 = F_1$ . Then, by a similar argument as in (1),  $x_i = x_{i+l}$  and so l = 0, which is a contradiction.
- 4. Suppose  $G_2 = F_2$ . We have either  $x_j = x_{i+j}$  or  $x_j = x_l$ . The first case yields i = 0 and thus a contradiction, so we conclude j = l.

We see that if neither  $G_1$  nor  $G_2$  are distinct equations, then i = j = l (and by extension, i + j = k = 2i) and so we have a tripling ridge. So the theorem holds.

**Example 4.10.** See Figure 4.2, in which we show poset diagrams for an esker ridge, a tripling ridge, and a nonridge of  $C_7$ , with their intersecting facet equations. Notice that, due to transitivity mentioned in the above proof, the nonridge in 4.2c has the additional relations  $1 \leq 2$  and  $5 \leq 2$  yielding the additional equations  $2x_1 = x_2$  and  $x_4 + x_5 = x_2$  (and the final additional relation  $4 \leq 2$ ).

From the Thereford 4.9, we are able to determine, given any two facets of a Kunz cone, whether they intersect in a ridge or in a face of codimension > 2. Finally, we present a formula for the number of ridges in any Kunz cone. We first must count the number of facets.

<sup>&</sup>lt;sup>3</sup>Note that if  $m \leq 4$  and there is collapse, we may have i + j + l = i and so this additional equation does not actually exist since y must be  $x_0$ , and subscripts of 0 are not allowed.



Figure 4.2. Posets for an esker ridge, a tripling ridge, and a nonridge of  $C_7$ .

**Lemma 4.11.**  $C_2$  has one facet, the vertex. For  $m \ge 3$ , the number of facets is given by

$$\bar{F} = \begin{cases} \frac{m(m-2)}{2} & \text{if } m \equiv 0 \mod 2\\ \frac{(m-1)^2}{2} & \text{if } m \equiv 1 \mod 2 \end{cases}.$$
(4.9)

*Proof.* Let the facet equation be given by  $F: x_i + x_j = x_{i+j}$ . We first choose i; there are m - 1 values i can take. We next choose i + j; there are m - 2 values i + j can take since  $i + j \in \{1, \ldots, m - 1\} \setminus \{i\}$  and j is uniquely determined by this choice. These (m - 1)(m - 2) choices count each F with distinct i, j twice, but each F with i = j just once. We can double count each F with i = j and then divide by 2 to get the total number of facets. If  $2 \mid m$ , then  $i \in \{1, \ldots, m - 1\} \setminus \{\frac{m}{2}\}$ , so there are m - 2 equations  $2x_i = x_{2i}$ , so we have

$$\frac{(m-1)(m-2) + (m-2)}{2} = \frac{m(m-2)}{2}$$

If  $2 \nmid m$ , there are m - 1 equations  $2x_i = x_{2i}$ , so we have

$$\frac{(m-1)(m-2) + (m-1)}{2} = \frac{(m-1)^2}{2}.$$

Together, we have the equation above.

We can now count the number of ridges.

**Theorem 4.12.** Let  $\overline{F}$  represent the number of facets of the Kunz cone. We also let  $\delta_q$  be the number of quadribigeminal faces, given in (4.1), and  $\delta_t$  be the number of tripling ridges, given in (4.8). For  $m \geq 5$ , the number of ridges of  $C_m$  is

$$\binom{\bar{F}}{2} - \left[\bar{F}(m-2) - \frac{(m-1)(m-2)}{2}\right] - \delta_q + \delta_t.$$
(4.10)

*Proof.* We begin with  $\binom{\bar{F}}{2}$ , taking every possible pair of intersecting facets. We remove each pair in which, without loss of generality, i + j = k. These are all the faces that are not esker faces. For any facet  $F_1 : x_i + x_j = x_{i+j}$ , the intersecting facet will be  $F_2 : x_{i+j} + x_k = x_{i+j+k}$ . Since  $i + j + k \neq 0$ ,  $k \in \{1, \ldots, m-1\} \setminus \{m - (i+j)\}$ . Thus there are m - 2 choices for k and so we have counted  $\bar{F}(m-2)$  intersections. However, intersections in which collapse occurs (for example, when we have  $i \leq k$  and  $k \leq i$ ), have been counted twice. So we count the number of intersections of two facets which result in collapsed faces. Our equations are of the form

$$F_1: x_i + x_j = x_{i+j}$$
  
 $F_2: x_{i+j} + x_{-j} = x_i.$ 

By choosing i and i + j, we have a distinct value j that satisfies the equations. So we have (m-1) choices for i and (m-2) choices for i + j (since it must not be i or 0). We then divide by 2 since the choices of i and i + j are not ordered. So there are  $\bar{F}(m-2) - \frac{(m-1)(m-2)}{2}$  possible intersections which are not esker faces. Subtracting from  $(\frac{\bar{F}}{2})$ , we have the number of esker faces.

Lemma 4.5 tells us that quadribigeminal faces are the only esker faces that are not ridges, so we subtract  $\delta_q$ . Tripling ridges are not esker faces so they are not included in the count up to this point, but they are ridges and so must be added back in, so we add  $\delta_t$ . We are left with exactly the number of ridges.

To obtain a quasipolynomial for the number of ridges based solely on m modulo 12, we substitute the equations for  $\overline{F}$ ,  $\delta_t$ , and  $\delta_q$  into (4.10) and obtain:

$$\bar{R} = \begin{cases} \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - \frac{9}{4}m - 2 & \text{if } m \equiv 0 \mod 12 \\ \frac{1}{8}m^4 - m^3 + 3m^2 - 3m + \frac{7}{8} & \text{if } m \equiv 1 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - 2m - 1 & \text{if } m \equiv 2 \mod 12 \\ \frac{1}{8}m^4 - m^3 + 3m^2 - 3m - \frac{9}{8} & \text{if } m \equiv 3 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - \frac{9}{4}m & \text{if } m \equiv 4 \mod 12 \\ \frac{1}{8}m^4 - m^3 + 3m^2 - 3m + \frac{7}{8} & \text{if } m \equiv 5 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - 2m - 3 & \text{if } m \equiv 6 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - \frac{9}{4}m & \text{if } m \equiv 7 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - \frac{9}{4}m & \text{if } m \equiv 8 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - 2m - 1 & \text{if } m \equiv 9 \mod 12 \\ \frac{1}{8}m^4 - m^3 + \frac{11}{4}m^2 - 2m - 1 & \text{if } m \equiv 10 \mod 12 \\ \frac{1}{8}m^4 - m^3 + 3m^2 - 3m + \frac{7}{8} & \text{if } m \equiv 11 \mod 12 \end{cases}$$

As an example, we present Table 4.1 which shows every possible intersection of two facets in  $C_5$ . They are categorized into ridges and nonridges, and within ridges, categorized into tripling ridges and esker ridges. There are

$$\frac{5^4}{8} - 5^3 + 3(5)^2 - 3(5) + \frac{7}{8} = 14$$

ridges.

Having described and counted every ridge, we look towards future research.

	Ridges	Nonr	idges	
Tripling	Es	ker		
$2x_1 = x_2$				
$x_1 + x_2 = x_3$	$x_1 + x_3 = x_4$	$x_3 + x_4 = x_2$	$2x_2 = x_4$	$x_2 + x_4 = x_1$
$2x_2 = x_4$	$2x_1 = x_2$	$x_1 + x_2 = x_3$	$2x_1 = x_2$	$x_1 + x_2 = x_3$
$x_2 + x_4 = x_1$	$2x_4 = x_3$	$2x_2 = x_4$	$2x_3 = x_1$	$x_1 + x_3 = x_4$
$2x_3 = x_1$	$x_1 + x_2 = x_3$	$2x_2 = x_4$	$x_1 + x_2 = x_3$	$x_1 + x_2 = x_3$
$x_1 + x_3 = x_4$	$2x_4 = x_3$	$2x_3 = x_1$	$x_2 + x_4 = x_1$	$2x_3 = x_1$
$2x_4 = x_3$	$x_1 + x_3 = x_4$	$x_2 + x_4 = x_1$	$x_1 + x_2 = x_3$	$x_1 + x_3 = x_4$
$x_3 + x_4 = x_2$	$2x_2 = x_4$	$2x_3 = x_1$	$x_3 + x_4 = x_2$	$x_2 + x_4 = x_1$
	$x_2 + x_4 = x_1$	$2x_3 = x_1$	$x_1 + x_3 = x_4$	$x_1 + x_3 = x_4$
	$2x_4 = x_3$	$x_3 + x_4 = x_2$	$x_3 + x_4 = x_2$	$2x_4 = x_3$
			$2x_2 = x_4$	$2x_2 = x_4$
			$x_3 + x_4 = x_2$	$2x_4 = x_3$
			$x_2 + x_4 = x_1$	$2x_3 = x_1$
			$x_3 + x_4 = x_2$	$2x_4 = x_3$

Table 4.1. Every intersection of two facets in  $C_5$ .

## CHAPTER 5 Summary and Future Research

In this thesis we have presented the full face structure of  $C_5$  both with a face lattice and with a labeled cross section. We have presented the full face structure of  $C_6$ with a Schlegel projection of a cross section, highlighting the symmetry present. We have presented the face structure of a representative of each type of facet present in  $C_7$ . We have also presented a theorem to tell when, given two facets of a Kunz cone, the intersection is a ridge or is a face of lower dimension. Lastly, we have provided a formula to count the number of ridges of  $C_m$  for  $m \ge 5$ . Many questions remain, and there is plenty open for future research, including:

- How do the  $C_7$  facets intersect with each other? Do each type of facet intersect with each other type, or do some types only intersect with one of the others? How many dimensions do these intersections have? How do these 18 facets fit together to form  $C_7$ ?
- When does the irregular gyrobifastigium shape appear? We have found it in  $C_5$  and  $C_7$ , and a split gyrobifastigium (two triangular prisms that together make a gyrobifastigium) appears in  $C_6$  (the outside faces). Do these occur frequently in higher dimensional Kunz cones? Are they only in  $C_m$  with m under some divisibility restrictions?
- Are there any 3-dimensional faces involving more than 4 rays? Previous work done by Joe McDonough and Cole Brower suggests not, but it has yet to be proved.
- Is there a limit on the number of rays involved in 4-dimensional faces? In the ones studied in this thesis, none involve more than 8 rays, but there are 9-ray faces in  $C_9$  and  $C_{10}$ , though counts for m > 10 were not able to be determined with the available computers. If a limit exists for 3-dimensional faces, it is plausible that a limit may exist for each dimension.
- Is it possible to find a face-regular gyrobifastigium in any Kunz cone?  $\hat{C}_5$  has been determined not to be face-regular at a fixed coordinate sum, but could one appear in a higher dimensional Kunz cone?

We hope this thesis serves as a building block to support future research on the family of Kunz cones.

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## APPENDIX

# 3-DIMENSIONAL CHAMBERS OF THE SCHLEGEL PROJECTION OF $\hat{C}_6$

## 3-DIMENSIONAL CHAMBERS OF THE SCHLEGEL PROJECTION OF $\hat{C}_6$

We show ten of the twelve internal 3-dimensional chambers of the Schlegal projection of  $\hat{C}_6$ . Facets are shown in symmetric pairs and each pair is shown at two different angles to help with visualization. The two outside facets are not shown. These images were made with SageMath.



