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1980

2 paper

#41 1981

## Problems Drive 1980

Questions

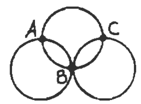
1. Find all real solutions of:
   (i) $\sin(60^{\circ}+x) - \sin(60^{\circ}-x) = \sin x$ ;
   (ii) $\cos^2 x - \frac{1}{4} = \cos(x+30^{\circ})\cos(x-30^{\circ})$;
   (iii) $1 = \cos 4x - \tan 43x \sin 4x$ .

2. Prove that if a,b,c are integers such that a+b+c is even then there is an integer n such that

   ab+n, ac+n, bc+n are all perfect squares.

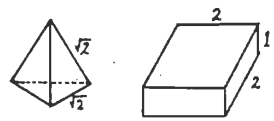
   Be brief!

3. Three towns A,B,C are connected by seven paths as shown in the diagram.

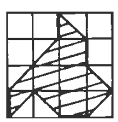



   A group of Archimedeans starting at A decides to go on a ramble round these paths in such a way that they travel along each of the seven paths exactly once.
   In how many ways can this be done?

4. How many regular tetrahedra with sides of length $\sqrt{2}$ can be fitted into a box with inside dimensions of 1×2×2 ?




5. Dissect this shape into three congruent (allowing reflections) pieces:



6. What is the largest real number x for which there exist four different numbers in the interval [0,1] (i.e. $0 \le x_i \le 1$) such that for any three of them which are not all equal

   $$|x_0+x_1-2x_2| \ge x \quad .$$

7. The following conversation was overheard at a maths. lecture where all mathematicians were either Pure or Applied:
   A: I am Pure and so is C .
   B: C is not Pure.
   C: B is Pure or A is Applied.
   Given that Pure mathematicians always tell the truth and Applied mathematicians always lie, what were A,B, and C; or is it impossible to tell? All three are mathematicians.

8. This 4×4 grid used to have a letter written in each square. The 8 "words" spelt across and down were:
   SATT SSAT STAS STSA
   ATST ATAS TAST TATA .
   What "word was spelt down the diagonal from top left to bottom right?

9. If $\alpha$ is a root of the cubic equation:
   $$x^3 - x^2 + x + 1 = 0 ,$$
   then of what cubic equation with integer coefficients is $\alpha^2$ a root?

10. What is the next number in each of these sequences?
   (i) 134, 93, 41, 11, 8, 3 ;
   (ii) 1, 3, 25, 253, 3121 ;
   (iii) 00000, 30103, 47712, 60206, 69897 ;
   (iv) 2, 6, 20, 70, 252 .

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11. Give an example of a cubic polynomial $P(x) = ax^3+bx^2+cx+d$ $(a \ne 0)$ with integer coefficients such that P(0), P(1), P(2), P(3) and P(4) are all perfect squares of integers.

12. Solve in integers x & y, $(x+y)^2-(x+2)^2-(y+1)^2 = 1$ .

Then consider the operations that replace

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \quad \text{by} \quad \begin{array}{ccc} g & d & a \\ h & e & b \\ i & f & c \end{array}, \quad \begin{array}{ccc} wc & a & b \\ f & d & e \\ \bar{w}i & g & h \end{array}, \text{ and } \begin{array}{ccc} g+x & b+y & i+x \\ f+y & e & d+y \\ a+x & h+y & c+x \end{array} \quad \begin{pmatrix} x = a+c+g+i \\ y = b+d+f+h \end{pmatrix}$$

which I call C, E and H because they are a Clockwise turn, Eastward shift (with multipliers) and an H-shaped permutation (with additions).

We suspect, but haven't proved, that these generate a group which, if we work modulo scalar multiplication by w, is Janko's third group $J_3$, of order 50,232,960. Can you prove this?

It suffices to show that it is not possible to find a sequence of these operations that will take you from

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \quad \text{to} \quad \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{array} .$$

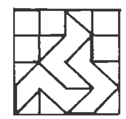
(We know that every vector with an odd number of non-zero coordinates can be taken to one of these, and that all non-zero vectors with evenly many non-zero coordinates are equivalent.)

It's fun playing with these operations, and surely someone must come up with the easy proof we have missed! Write to me when you do, and we'll publish it. Good luck!



# Problems Drive

Answers

1. (i) All real x.
   (ii) No real x .
   (iii) $x = n(4^o)$ , n integer. Note that $x = n(90^o)$ does not work for odd n and coincides with the given solution for n even.
2. By example: $4n = a^2+b^2+c^2-2ab-2ac-2bc$
   or $n = \frac{1}{4}/a+b-c)^2 - ab$ (etc.) .
3. 288 .
4. 5 .
5. 
6. $\frac{1}{4}$ (i.e. $(o,\frac{1}{4},\frac{1}{4},1)$).
7. A Applied
   B Applied
   C Pure .
8. SAAT, e.g.

| | | | |
|---|---|---|---|
| STAS | STSA | SSAT | STSA |
| TATA | TAST | TATA | SATT |
| SSAT | ATAS | STAS | ATAS |
| ATST | SATT | ATST | TAST . |

9. $n(x^3+x^2+3x-1) = 0$ .
10. (i) 2 (Euclid algorithm).
    (ii) 46651 $(= 6^6 - 5)$ .
    (iii) 77815 $(= 10^5 \log_{10} 6$, to the nearest integer).
    (iv) 924 $\left(= \frac{12!}{6!\,6!}\right)$ .
11. e.g.
$$\begin{aligned} 6x(x-1)(x-2) &= 6x^3-18x^2+12x \\ 4x(x-1)(x-2)+25 &= 4x^3-12x^2+8x+25 \\ 25+4(x-1)(x-2)(x-3) &= 4x^3-24x^2+44x+1 \\ -6(x-2)(x-3)(x-4) &= -6x^3+54x^2-156x+144 \\ 25-4(x-2)(x-3)(x-4) &= -4x^3+36x^2-104x-121 \\ 25-4(x-1)(x-2)(x-3) &= -4x^3+24x^2-44x+49 \\ 25\pm 8x(x-2)(x-4) &= \pm 8x^3 \mp 48x^2 \pm 64x+25 . \end{aligned}$$
12. (x,y) = (6,3)
    (2,7)
    (0,-3)
    (-4,1)