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ASYMPTOTIC NUMBER OF SELF-CONVERSE ORIENTED GRAPHS

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[Rob77]

ABSTRACT

The exact number $\bar{O}_p$ of self-converse oriented graphs on p points has been known for some time; we give the first asymptotic analysis of $\bar{O}_p$. The formula for $\bar{O}_p$ was found by an application of Burnside's Lemma, and consists of a sum of terms corresponding to the partitions of p. The usual pattern in such cases is for one term to dominate the others when p is large. For $\bar{O}_p$ there is no single term which is dominant. Instead an infinite family of terms must all be considered in the asymptotic treatment. This leads to a much more complicated asymptotic form for $\bar{O}_p$ than the ones obtaining for graphs, digraphs, self-complementary graphs, self-complementary digraphs, oriented graphs, tournaments, or self-complementary tournaments. The values of $\bar{O}_p$ are tabulated for p up to 27. A sample is then compared with the corresponding sums of dominant terms and final asymptotic expressions.

1. INTRODUCTION

An *oriented graph* is a finite digraph with no symmetric pairs of arcs. Thus an oriented graph has no loops and at most one arc joining any unordered pair of points. The *converse* of any digraph is obtained by reversing the directions of all of its arcs. A digraph is *self-converse* just if it is isomorphic to its converse.

The number $\bar{O}_p$ of nonisomorphic self-converse oriented graphs on p points was found by Sridharan [9] using Burnside's Lemma. His result can be put in the form

$$\bar{O}_p = \sum 3^{c\langle h\rangle} \Big/ \prod_i i^{h_i} h_i! , \tag{1}$$

the sum being over all sequences $\langle h\rangle = (h_1, h_2, \ldots)$ of non-negative integers with $\sum_{1\le i} ih_i = p$. The exponent $c\langle h\rangle$ is given by

$$c\langle h\rangle = \sum_{1\le i} (ih_{2i}^2 - h_{4i}) + \sum_{\substack{1\le i<\ell \\ i\ell \text{ even}}} (i,\ell) h_i h_\ell . \tag{2}$$

Here as usual (i,ℓ) denotes the greatest common divisor of i and ℓ.

In this paper we give the first asymptotic analysis of $\overline{O}_p$. It is found that

$$(3)\quad \overline{O}_p \sim 3^{p^2/4}\, p^{-p/2+(\log_3 p)/4-\log_3\log_3 p-1/2+\log_3 e}\, e^{p/2} (\log_3 p)^{\log_3\log_3 p-1/2} C_p$$

where the factor C_p is of constant order. The latter may be expressed in terms of the function Φ defined by

$$\Phi(x) = \frac{1}{\sqrt{\pi \log_3 e}} \sum_{k=-\infty}^{\infty} 3^{-(x-k)^2} .$$

Then (3) is completed by

$$(4)\qquad C_p = \sqrt{\frac{\log_3 e}{2\pi}}\, \Phi\left(\tfrac{1}{2}\log_3 p - \log_3\log_3 p - \tfrac{1}{2}\rho(p)\right)$$

where $\rho(p)$ denotes the residue of p modulo 2.

An unusual feature of the asymptotic behaviour of $\overline{O}_p$ is that as $p \to \infty$ infinitely many terms from the right side of (1) make contributions which cannot be neglected. The normal pattern is that a single term dominates the asymptotic behaviour in unlabelled graph counting problems to which the exact answer is obtained fairly directly using Burnside's Lemma. This has been shown to be the case for graphs by Oberschelp [8], for relational systems of uniform non-monodic type by Oberschelp [7], for all relational systems of non-monodoc type by Fagin [4], for digraphs by Harary and Palmer [2, p.200], for self-complementary graphs and digraphs by Palmer [6], for oriented graphs by Wille [10], for tournaments by Moon [5, p.88], and for self-complementary tournaments in [2, exercise 9.6].

The pattern of asymptotic behaviour shown here to hold for self-converse oriented graphs was first observed in a graphical context in the joint work [3] with Harary, Palmer and Schwenk. There similar asymptotic expressions are reported for the numbers of self-dual signed graphs and line-self-dual nets, but no details of the derivation are given. In fact they follow closely the development outlined herein for self-converse oriented graphs. The asymptotic number of self-converse digraphs also involves an infinite series of dominant terms, but in a sufficiently different fashion that a separate description is necessary; this will appear elsewhere.

We begin the proof of (3) and (4) in the next section by selecting the dominant terms in the right side of (1) and showing that the remainder can be neglected asymptotically. Then the sum of the dominant terms is evaluated asymptotically in Section 3. It is found that the cases p even and p odd must be treated

These are compared numerically with the corresponding sums of dominant terms and their asymptotic approximations.

2. DOMINANT TERMS

In this section we show that asymptotically it is sufficient to restrict attention to terms on the right side of (1) contributed by sequences <j> of the form (j_1,j_2), that is, those with $j_3 = j_4 = \ldots = 0$. We call these the *dominant terms*. In showing that the contribution of the other terms in (1) can be neglected, we compare the term due to an arbitrary $\langle h\rangle = (h_1,h_2,\ldots)$ with the dominant term given by

$$(5) \qquad j_1 = \begin{cases} h_1+1 & \text{if } p-h_1 \text{ is odd,} \\ h_1 & \text{if } p-h_1 \text{ is even} \end{cases} .$$

Here p refers to the *order* $p = \sum_{1\le i} ih_i$ of <h>. The dominant term to which we compare <h> is understood to be of the same order, so that

$$(6) \qquad j_2 = \frac{p-j_1}{2}.$$

For each sequence <h> let t<h> be the corresponding term in the expression (1) for $\overline{O}_p$ where p is the order of <h>; that is,

$$(7) \qquad t\langle h\rangle = 3^{c\langle h\rangle}/\prod i^{h_i}h_i! \; .$$

The power c<h> is simply the total number of ordinary (non-diagonal) line cycles of even length and diagonal line cycles of odd length induced by a point permutation which is the product of h_i disjoint cycles of length i over $i \ge 1$. The fraction $1/\prod i^{h_i}h_i!$ is the proportion of permutations on $\{1,2,\ldots,p\}$ which can be written as a product of this sort, out of the p! possible permutations.

For the time being we fix a dominant term $t(j_1,j_2)$ and consider all terms which are to be compared with it. Any such term t<h> corresponds to a sequence <h> satisfying one of two conditions for some integer $m \ge 0$;

$$(a) \qquad h_1 = j_1,\; h_2 = j_2-m \text{ and } \sum_{2<i} ih_i = 2m, \text{ or}$$

$$(b) \qquad h_1 = j_1-1, h_2 = j_2-m \text{ and } \sum_{2<i} ih_i = 2m+1.$$

In case (a) $m \ne 1$ and in case (b) $m \ne 0$, but otherwise all values of m up to j_2 are possible.

We now show how to calculate an upper bound on the sum of terms obtained from all <h> which are compared to <j> and satisfy (a) for a fixed $m \geq 0$. First we will show that

$$(8) \qquad c\langle h\rangle \leq j_2(j_2 - m) + \frac{m}{2}(m-1) + j_1(j_2 - \frac{m}{2}) .$$

The right side of (8) is the value of $c(j_1, j_2-m, 0, \frac{m}{2})$ computed from (2) regardless of whether $\frac{m}{2}$ is integral. The left side of (8) is given by (2); with the values j_1 for h_1 and j_2-m for h_2 from (a) and the terms in j_1 and j_2-m separated from the sums, (2) takes the form

$$(9) \qquad c\langle h\rangle = (j_2-m)^2 + \sum_{\substack{2<i \\ i \text{ even}}} \frac{i}{2} h_i^2 - \sum_{1\leq i} h_{4i} + j_1(j_2-m) + j_1 \sum_{\substack{2<i \\ i \text{ even}}} h_i$$

$$+ (j_2-m) \sum_{\substack{1<i \\ i \text{ odd}}} h_i + 2(j_2-m) \sum_{\substack{2<i \\ i \text{ even}}} h_i + \sum_{\substack{2<i<\ell \\ i\ell \text{ even}}} (i,\ell) h_i h_\ell .$$

Selecting two terms each from (9) and (8) we obtain the difference

$$j_2(j_2-m) + j_1(j_2-\frac{m}{2}) - (j_2-m)^2 - j_1(j_2-m),$$

which simplifies to

$$(j_1+2j_2-2m)\frac{m}{2} .$$

We make the replacement

$$\frac{m}{2} = \frac{1}{4} \sum_{\substack{2<i \\ i \text{ even}}} ih_i + \frac{1}{4} \sum_{\substack{1<i \\ i \text{ odd}}} ih_i$$

in view of (a). Now note that the result is greater than or equal to the sum

$$j_1 \sum_{\substack{2<i \\ i \text{ even}}} h_i + 2(j-m) \sum_{\substack{2<i \\ i \text{ even}}} h_i + (j_2-m) \sum_{\substack{1<i \\ i \text{ odd}}} h_i$$

contained in (9). This is because $\frac{i}{4} \geq 1$ for even $i > 2$ and $\frac{i}{2} > 1$ for odd $i > 1$.

Therefore to verify (8) it will suffice to show the inequality

$$\sum_{\substack{2<i\\ i\text{ even}}} \frac{i}{2} h_i^2 - \sum_{1\le i} h_{4i} + \sum_{\substack{2<i<\ell\\ i\ell\text{ even}}} (i,\ell)h_i h_\ell \le \frac{m^2}{2} - \frac{m}{2}$$

among the remaining terms from (8) and (9). By (a) again we may replace $\frac{m^2}{2} - \frac{m}{2}$ by

$$\frac{1}{8} \sum_{2<i} i^2 h_i^2 + \frac{1}{4} \sum_{2<i<\ell} i\ell h_i h_\ell - \frac{1}{4} \sum_{2<i} i h_i,$$

and upon rearranging obtain the equivalent inequality

$$(10) \qquad \sum_{\substack{2<i\\ i\text{ even}}} \frac{i}{2} h_i^2 + \frac{1}{4} \sum_{2<i} i h_i + \sum_{\substack{2<i<\ell\\ i\ell\text{ even}}} (i,\ell)h_i h_\ell \le \frac{1}{8} \sum_{2<i} i^2 h_i^2 + \sum_{1\le i} h_{4i} + \frac{1}{4} \sum_{2<i<\ell} i\ell h_i h_\ell .$$

Now (10) is verified by observing that $(i,\ell) < \frac{i\ell}{4}$ when $2 < i < \ell$ and one of i,ℓ is even, $\frac{i}{2} = \frac{i^2}{8}$ and $\frac{i}{4} = 1$ when $i = 4$, $h_i \le h_i^2$ for all i (since h_i must be a non-negative integer), $\frac{i}{4} < \frac{i^2}{8}$ for odd $i > 2$, and $\frac{3}{4} i \le \frac{i^2}{8}$ for even $i > 4$.

Next, the number of all <h> which are compared to <j> and satisfy (a) for a particular $m \ge 0$ is bounded above by

$$(11) \qquad \binom{p}{j_1}\binom{p-j_1}{2j_2-2m}\frac{(2j_2-2m)!}{(j_2-m)!\,2^{j_2-m}}\,(p-j_1-2j_2+2m)! \quad .$$

This is derived by considering the number $\binom{p}{j_1}$ of ways to choose a set of j_1 fixed points from the object set $1,2,\ldots,p$ followed by the number $\binom{p-j_1}{2j_2-2m}$ of ways to choose a set of $2j_2-2m$ elements from the remainder. The latter can be arranged into j_2-m transpositions in exactly $(2j_2-2m)!/(j_2-m)!\,2^{j_2-m}$ ways, since the transpositions can be rearranged among themselves in $(j_2-m)!$ ways and have starting points chosen in 2 ways each for a total of 2^{j_2-m} ways altogether. There are then $p-j_1-2j_2+2m$ elements to be permuted among themselves with no fixed points or transpositions, and $(p-j_1-2j_2+2m)!$ gives a crude upper bound for the number of ways this can be done.

Finally, multiplying together the upper bound for $3^{c\langle h\rangle}$ from (8) with the upper bound (11) for the number of terms and then dividing by $p!$ gives

$$(12) \qquad \frac{3^{j_2(j_2-m) + (m-1)m/2 + j_1(j_2-m/2)}}{j_1!\,(j_2-m)!\,2^{j_2-m}} \quad .$$

This is the desired upper bound for the total contribution of the terms $t\langle h\rangle$ to $\overline{O}_p$ in the right side of (1), for all <h> satisfying (a) for fixed $m \ge 0$.

The terms bounded by (12) are all to be compared with the dominant term $t\langle j\rangle$, where

$$(13) \qquad t\langle j\rangle = \frac{3^{j_2(j_1+j_2)}}{j_1!j_2!2^{j_2}} \; .$$

Using the fact that $j_2!/(j_2-m)! \le j_2^m$, we see that the ratio of the upper bound (12) to the dominant term (13) is at most

$$(14) \qquad \left[\frac{2j_2}{3^{(j_1+2j_2-m+1)/2}}\right]^m .$$

Then, from $j_1+2j_2 = p$ and $m \le p/2$ it follows that the latter is less than

$$(15) \qquad \left(p3^{-p/4}\right)^m .$$

The next stage is to establish a similar upper bound for the sum of the terms $t\langle h\rangle$ contributed by sequences $\langle h\rangle$ which satisfy (b). If $\langle h\rangle$ satisfies (b) for some m, then $m \ge 1$ and the exponent $c\langle h\rangle$ obeys the inequality

$$(16) \qquad c\langle h\rangle \le j_1(j_2 - \tfrac{m+1}{2}) + (j_2-m)(j_2-1) + \frac{(m-1)(m-2)}{2}$$

when $p \ge 12$. The right side of (16) is the value of $c(j_1-1, j_2-m, 1, \frac{m-1}{2})$ computed from (2) whether or not $\frac{m-1}{2}$ is an integer. A proof of (16) can be carried out along the same lines as the proof of the similar inequality (8). It is not quite as neat in this case, for a consideration of $\langle h\rangle = (i,0,1,0,0,1)$ for $i = 0,1$ or 2 will verify that the condition $p \ge 12$ is essential.

The number of sequences $\langle h\rangle$ which are compared to $\langle j\rangle$ and satisfy (b) for a particular $m \ge 1$ is bounded by

$$(17) \qquad \binom{p}{j_1-1}\binom{p-j_1+1}{2j_2-2m}\frac{(2j_2-2m)!}{(j_2-m)!2^{j_2-m}}\,(p-j_1-2j_2+2m+1)! \; .$$

The reasoning is exactly the same as for the similar bound (11), the only difference being that now $h_1 = j_1-1$, so that j_1 has been replaced by j_1-1. Multiplying this together with the upper bound for $3^{c\langle h\rangle}$ obtained from (16) and dividing by p! gives

$$(18)\qquad \frac{3^{j_1(j_2-(m+1)/2)+(j_2-m)(j_2-1)+(m-1)(m-2)/2}}{(j_1-1)!(j_2-m)!2^{j_2-m}} .$$

This is an upper bound for the total of the terms t<h> for all <h> satisfying (b) for some $m \geq 1$ and fixed $\langle j\rangle = (j_1,j_2)$. Dividing by the value of t<j> given in (13) and again using the fact that $j_2!/(j_2-m)! \leq j_2^m$, we find that the ratio is less than or equal to

$$(19)\qquad j_1 3^{-j_1/2-j_2+1}\left(2j_2 3^{-j_1/2-j_2+(m-1)/2}\right)^m .$$

From the relations $j_1+2j_2 = p$ and $m < p/2$ it follows that (19) is less than

$$(20)\qquad \left(p3^{-p/4}\right)^{m+1}$$

for sufficiently large p.

We now collect each dominant term in the equation (1) for $\overline{O}_p$ together with all the other terms which are compared to it. These will satisfy (a) for $m \geq 2$ or (b) for $m \geq 1$; so by summing the upper bounds (15) and (20) over these values of m we find that the total contribution of the terms compared to $t(j_1,j_2)$ is

$$t(j_1,j_2)\left(1+O(p^2 3^{-p/2})\right) .$$

Since all of these terms are positive and the upper bounds used are independent of the particular dominant term, we may sum over all dominant terms to obtain

$$(21)\qquad \overline{O}_p = \left(\sum_{m\leq p/2} t(p-2m,m)\right)\left(1+O(p^2 3^{-p/2})\right) .$$

Thus we may neglect the nondominant terms of $\overline{O}_p$ asymptotically. The rapid convergence of the sum of the dominant terms to $\overline{O}_p$ is illustrated later in Table 2 for selected values of p up to 27.

3. ASYMPTOTIC FORMULAE

In this section we discuss the asymptotic behaviour of the sum of the dominant terms t(p-2m,m). A general approach to the asymptotic evaluation of sums with positive terms is described by Bender [1, Section 3]. Two atypical features appear in the process of following the usual approach. One is that the result depends on whether p is even or odd. The other is that the factor of constant order contains a periodic function of $\frac{1}{2}\log_3 p - \log_3\log_3 p$.

Consider first the case of even p, and let $p = 2n$. Denote by $t(k)$ the dominant term $t(2k, n-k)$. Then from (13) we have

$$(22) \qquad t(0) = 3^{n^2}/n!2^n$$

and

$$(23) \qquad t(k)/t(0) = n!2^k/(2k)!(n-k)!3^{k^2}.$$

The ratio of successive terms is

$$(24) \qquad t(k+1)/t(k) = (n-k)/(k+1)(2k+1)3^{2k+1}.$$

From this it is not hard to calculate that $t(k)$ should attain its maximum value for k near to $f(n)$, where

$$(25) \qquad f(n) = \tfrac{1}{2}\log_3 n - \log_3\log_3 n + \tfrac{1}{2}\log_3 2.$$

To study $t(k)$ in the vicinity of its maximum value, let $x = k - f(n)$ and restrict attention to the region $|x| \leq (\log_3 n)^{\frac{1}{4}}$. Using Stirling's formula for the factorial functions in (23) it is straightforward to calculate

$$(26) \qquad t(k)/t(0) = \frac{(2n)^{f(n)}3^{-x^2}}{\{2f(n)\}!3^{f(n)^2}} \left(1 + O(\log\log n/\sqrt{\log n})\right).$$

Here the implied constant in the error term can be taken independent of x and n, and $\{2f(n)\}!$ is defined in terms of the Γ function.

From (24) it can be seen that the ratio $t(k+1)/t(k)$ is a unimodal function of k. In the region $|x| \leq (\log_3 n)^{\frac{1}{4}}$ it is easy to calculate that $t(k+1)/t(k)$ is of order 3^{-2x}. Thus we have

$$(27) \qquad \sum_{k=0}^{n} t(k) = \left(\sum{}' t(k)\right)\left(1 + O(\exp(-\sqrt{\log n}))\right),$$

where Σ' denotes summation over the integers k such that $|k - f(n)| \leq (\log_3 n)^{\frac{1}{4}}$. On the other hand

$$(28) \qquad \sum_{k=-\infty}^{\infty} 3^{-(f(n)-k)^2} = \left(\sum{}' 3^{-(f(n)-k)^2}\right)\left(1 + O(\exp(-\sqrt{\log n}))\right)$$

also, so that in approximating the right side of (27) using (26) the sum can be extended over all k. The resulting expression for the sum of the dominant terms is

$$(29) \qquad \sum_{k=0}^{n} t(k) = \frac{(2n)^{f(n)} t(0)}{(2f(n))!\,3^{f(n)^2}} \sum_{k=-\infty}^{\infty} 3^{-(f(n)-k)^2} (1+E(n))$$

where $E(n) = O(\log\log n/\sqrt{\log n})$.

In the terminology introduced for equation (4) the summation on the right side of (29) is

$$\sqrt{\pi \log_3 e}\ \Phi(f(n)).$$

The asymptotic formula for $\overline{O}_p$ given in (3) and (4) for even p can now be calculated by combining (21), (22), (25) and (29), eliminating the factorials with Stirling's formula and expressing the result in terms of p. The order of the error term obtained is $\log\log p/\log p$.

The derivation of the asymptotic behaviour of $\overline{O}_p$ for odd p is quite similar to that for even p. If $t^*(k)$ denotes the dominant term $t(2k+1, n-k)$ for $p = 2n+1$, the maximum is attained for k near to

$$(30) \qquad f^*(n) = \tfrac{1}{2}\log_3 n - \log_3\log_3 n + \tfrac{1}{2}\log_3 2 - \tfrac{1}{2}.$$

There are a number of minor differences between the expressions for $\overline{O}_{2n}$ and $\overline{O}_{2n+1}$ in terms of n. These cancel out asymptotically when the formulae are put in terms of p, all except for the respective periodic factors $\Phi(f(p/2))$ and $\Phi(f^*((p-1)/2))$.

From the definition it is evident that Φ is a periodic function, with period 1 and average 1. This is one of a class of functions considered by Wright [11] in connection with the asymptotic number of labelled k-coloured graphs. With 3 replaced by 2 as the base, Wright showed that this function varies very little from unity. The same method of proof would apply to our function $\Phi(x)$ in which 3 is the base. Numerical calculation indicates that the extreme values of $\Phi(x)$ differ by no more than 0.5×10^{-3}.

In the next section numerical values of the asymptotic expression are compared with the exact values of $\overline{O}_p$ for selected $p \leq 27$. No tendency toward convergence of the two is evident. As will be seen, much higher values of p would have to be used in order to observe the convergence which must eventually occur.

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$\bar{O}_p$	p
1	1
2	2
5	3
18	4
102	5
848	6
12 452	7
265 759	8
10 454 008	9
598 047 612	10
63 620 448 978	11
9 974 635 937 844	12
2 905 660 724 913 768	13
1 268 590 412 128 132 389	14
1 023 130 650 177 394 611 897	15
1 258 149 993 547 327 488 275 562	16
2 834 863 110 716 120 144 290 954 314	17
9 900 859 865 505 110 360 978 721 901 778	18
62 789 966 700 541 818 490 820 660 260 219 085	19
626 754 770 688 026 263 598 465 802 369 258 636 679	20
11 256 824 990 452 447 883 980 190 391 781 422 490 449 643	21
322 709 197 224 240 210 184 021 932 827 114 034 505 695 268 296	22
16 500 229 648 971 657 958 448 757 393 252 063 495 966 875 945 342 103	23
1 363 854 423 811 897 643 093 117 996 577 197 078 873 109 190 304 984 874 334	24
199 403 043 497 277 732 688 200 817 094 827 202 121 041 725 841 248 825 841 470 489	25
47 673 458 041 008 308 339 959 667 197 303 459 424 735 715 919 978 766 546 406 524 143 862	26
20 006 994 424 095 895 824 681 312 306 066 615 139 962 829 403 824 584 051 878 040 528 717 145 556	27

TABLE 1.

Exact numbers of self-converse oriented graphs for $p \leq 27$

4. NUMERICAL RESULTS*

The exact values of the numbers $\overline{O}_p$ of self-converse oriented graphs for $p \leq 27$ are presented in Table 1. These were produced by computer computation using a program which implemented equations (1) and (2) as directly as possible. In Table 2 are shown the results of comparing these numbers to the sum D_p of dominant terms and the final asymptotic formula A_p obtained by combining the right sides of equations (3) and (4). These are shown for sample values of p.

It will be seen that D_p approaches $\overline{O}_p$ rapidly and evenly as p increases. The same certainly cannot be said of A_p for the values of p considered. Since it was impractical to carry the exact computation of $\overline{O}_p$ significantly further, numerical testing was conducted on the various steps taken in deriving A_p as an asymptotic expression for D_p. One particular step with a $\log\log p/\log p$ error term was observed to contribute the major part of the apparently divergent behaviour for small p. Computation showed that convergent behaviour for this step set in at values of $\log_3 p$ between 30 and 40. Thus $A_p/\overline{O}_p$ cannot be expected to start decreasing as p increases, until p reaches 10^{15} or so. By that time $\overline{O}_p$ would have almost 10^{30} digits in its decimal representation.

$(\overline{O}_p/D_p) - 1$	$A_p/\overline{O}_p$	p
0.538×10^{-2}	1.624	8
0.227×10^{-2}	1.546	9
0.245×10^{-4}	1.701	14
0.956×10^{-5}	1.631	15
0.732×10^{-7}	1.741	20
0.272×10^{-7}	1.697	21
0.176×10^{-9}	1.768	26
0.636×10^{-10}	1.746	27

TABLE 2.

Comparison of exact numbers of self-converse oriented graphs with the sum of the dominant terms and the asymptotic formula.

* The computer programming for the reported data was performed by A. Nymeyer. The author is grateful to the Australian Research Grants Committee for its financial support, which provided technical assistance and some of the computing equipment used in this research.

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