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ON A GENERALIZATION OF PYTHAGORAS' THEOREM

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Pythagoras (*c.* 520 B.C.) has shown that the areas a^2, b^2 and c^2 constructed on the sides a, b and hypotenuse c of a right triangle satisfy the formula

$$c^2 = a^2 + b^2 \tag{1}$$

for all a, b, c.

In this note we shall consider six sequences of squares: a_i, b_i, c_i, x_i, y_i, z_i, $i = 1,2, \ldots$ constructed from the Pythagorean diagram; $a_1 = a$, $b_1 = b$, $c_1 = c$. (For compact notation we refer to each square by the length of its edge: 'Square b' means, e.g., 'the square of edge b'.) For $j = 1,2,3, \ldots$

- x_j is constructed on the segment joining adjacent corners of b_j and c_j
- y_j is constructed on the segment joining adjacent corners of a_j and c_j
- z_j is constructed on the segment joining adjacent corners of b_j and a_j

For $j = 2,3, \ldots$

- a_j is constructed on the segment joining adjacent corners of y_{j-1} and z_{j-1}
- b_j is constructed on the segment joining adjacent corners of x_{j-1} and z_{j-1}
- c_j is constructed on the segment joining adjacent corners of x_{j-1} and y_{j-1}

In the above definitions the 'adjacent corners' referred to are those 'exterior to the diagram' (see Fig. 1), which are not already connected, and such that the connection would not pass through the figure.

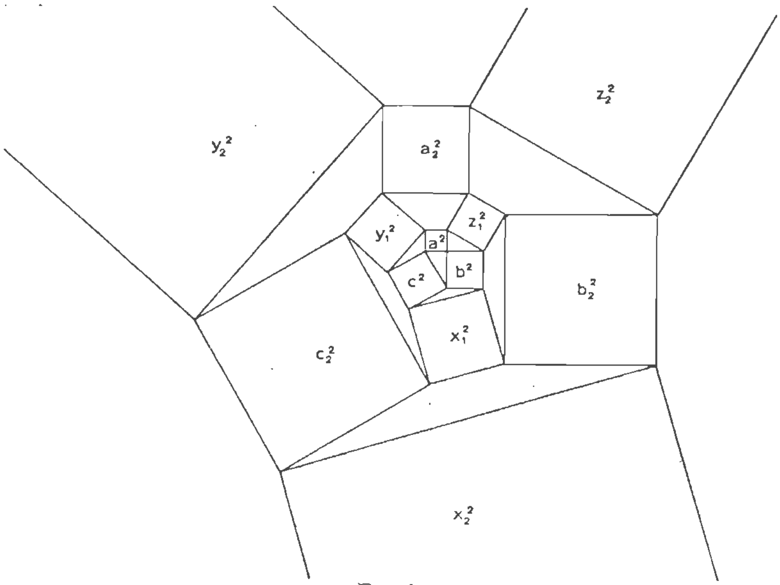

FIG. 1

Let the angle formed by a and c be arccos B
Let the angle formed by b and c be arccos A
Let the angle formed by a and y_1 be arccos α
Let the angle formed by c and y_1 be arccos β
Let the angle formed by c and x_1 be arccos γ
Let the angle formed by b and x_1 be arccos δ

Then

$$\begin{aligned} \arccos \alpha + \arccos \beta &= \arccos B \\ \arccos \gamma + \arccos \delta &= \arccos A \end{aligned} \tag{2}$$

and therefore

$$\begin{aligned} x_1{}^2 &= 3b^2 + c^2 \\ y_1{}^2 &= 3a^2 + c^2 \end{aligned} \tag{3}$$

and

$$x_1{}^2 + y_1{}^2 = 5z_1{}^2 \tag{4}$$

We can now express α, β, γ, δ easily in terms of known quantities:

$$\begin{aligned} \alpha &= 2a/y_1 \\ \beta &= (c^2 + a^2)/cy_1 \\ \gamma &= (b^2 + c^2)/cx_1 \\ \delta &= 2b/x_1 \end{aligned} \tag{5}$$

We note furthermore that

arccos α is the acute angle between each pair $a_i\,y_j$, $i = j, j+1, j = 1,2,3, \ldots$
arccos β is the acute angle between each par $c_i\,y_j$, $i = j, j+1, j = 1,2,3, \ldots$
arccos γ is the acute angle between each pair $c_i\,x_j$, $i = j, j+1, j = 1,2,3, \ldots$
arccos δ is the acute angle between each pair $b_i\,x_j$ $i = j, j+1, j = 1,2,3, \ldots$
arccos A is the acute angle between each pair $b_i\,z_j$ $i = j, j+1, j = 1,2,3, \ldots$
arccos B is the acute angle between each pair $a_i\,z_j$ $i = j, j+1, j = 1,2,3, \ldots$

Therefore the areas are connected by the recursive formulas

$$\begin{cases} a_j = a_{j-1} + Bz_{j-1} + \alpha y_{j-1} \\ b_j = b_{j-1} + \delta x_{j-1} + Az_{j-1} \\ c_j = c_{j-1} + \beta y_{j-1} + \gamma x_{j-1} \end{cases} \qquad j = 2,3, \ldots \qquad (6)$$

and

$$\begin{cases} x_j = x_{j-1} + \delta b_j + \gamma c_j \\ y_j = y_{j-1} + \alpha a_j + \beta c_j \\ z_j = z_{j-1} + Ab_j + Ba_j. \end{cases} \qquad j = 1,2, \ldots \qquad (7)$$

We find for example the coefficients shown in Table 1, where each entry gives the edge of the square shown on its left in multiples of the basic square shown at the head of its column.

	a	b	c	x	y	z
1	1	1	1	1	1	1
2	4	4	4	5	5	5
3	19	19	19	24	24	24
4	91	91	91	115	115	115

TABLE 1

It is clear that the general behaviour of a_j, b_j and c_j is identical and that x_j, y_j and z_j also increase equally. Henceforth, therefore we shall consider only a_j and x_j. We note in passing, however, that the Pythagorean relation (1) has generalizations

$$a_j^2 + b_j^2 = c_j^2, \; j = 1,2,3, \ldots \qquad (8)$$

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and that the corresponding formula (4) generalizes to

$$x_j^2 + y_j^2 = 5z_j^2, j = 1,2,3, \ldots. \tag{9}$$

Table 1 suggests the difference equation

$$a_{n+2} - 5a_{n+1} + a_n = 0 \tag{10}$$

which can be verified by mathematical induction on equations (6) and (7). The characteristic equation for (10) is [1, p. 548]

$$r^2 - 5r + 1 = 0 \tag{11}$$

from which we deduce that

$$a_n = Cp^n + Dq^n \tag{12}$$

where

$$C = \frac{7 - \sqrt{21}}{14} a \quad , \quad D = \frac{7 + \sqrt{21}}{14} a \tag{13}$$

and

$$p = \tfrac{1}{2}(5 + \sqrt{21}) \quad , \quad q = \tfrac{1}{2}(5 - \sqrt{21}). \tag{14}$$

After some calculations, it is possible to express the generating function

$$\pi(s) = \sum_{j=1}^{\infty} s^j a_j$$

in the form

$$\pi(s) = \frac{(1 - s)(1 + 5s + s^2)}{1 - 23s^2 + s^4} as. \tag{15}$$

The corresponding calculations for the x_n series are similar in principle and somewhat easier in practice. The difference equation

$$x_{n+3} - 6x_{n+2} + 6x_{n+1} - x_n = 0 \tag{16}$$

leads to the characteristic equation

$$(r - 1)(r^2 - 5r + 1) = 0. \tag{17}$$

The expression for x_n is

$$x_n = \frac{1}{\sqrt{21}}\left[\tfrac{1}{2}(5+\sqrt{21})\right]^n - \frac{1}{\sqrt{21}}\left[\tfrac{1}{2}(5-\sqrt{21})\right]^n \tag{18}$$

with generating function

$$\psi(s) = \sum_{j=1}^{\infty} s^j x_j = \frac{s x_1}{1 - 23s^2 + s^4} \tag{19}$$

I wish to express my gratitude to H. G. Forder; during our professional association between 1949 and 1957 he showed me that geometry is not yet exhausted.

Reference

1. Charles Jordan, *Calculus of Finite Differences*, 3rd ed., Chelsea, New York, 1965.