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MATHEMATICAL RECREATIONS by Ian Stewart

The Riddle of the Vanishing Camel

The Bedouin patriarch Mustapha ibn Mokhta successfully defended his small tribe against a fierce rival, thanks to Allah. But Mustapha had been mortally wounded in the fight and had fallen unconscious. His lifelong friend, Ali the barber, tended his wounds and then carried him for miles across the northern Arabian desert to their camp.

Mustapha awoke surrounded by his wives, sons, daughters and grandchildren. "Allah be praised, I am still alive. But I must get back to the fight." He could barely lift his head up.

"Please rest now, Great Mokhta," his first wife pleaded, offering him water from a goatskin. "You have led your tribe to victory. How do you feel?"

"Like I have been trampled by a thousand camels," Mustapha groaned. "Who saved me?"

"Ali the barber," his first wife replied.

"Bring him to me quickly." His first son left to summon Ali, and the family dispersed, leaving Mustapha to lie peacefully in his tent.

Ali was busy, as usual, trimming the beards of all Bedouins who do not trim their own and wondering who would trim his beard. Hearing that Mustapha had regained consciousness, he ran to visit his friend.

Ali entered Mustapha's tent. "Salaam aleikum. You look much better."

"Aleikum salaam. Thanks to you and Allah, I have had a chance to see my family again. Yet my body is broken beyond repair, and I fear I will die soon." He waved away the protest of his friend. "There is no need to pretend. I want to talk to you about how I should divide my wealth among my three sons. I am very fond of them, but they are sometimes slow-witted. I believe before they inherit anything they should demonstrate their intellectual prowess."

Ali looked perplexed. "I do not understand, Mustapha."

"Among my possessions is an ancient arithmetical treatise, handed down, it is said, from the great Al-Khowarizmi himself. It tells of a wealthy merchant who owned 17 camels. He decreed that on his death the eldest son was to have one half of the herd, the second son one third, and the third son one ninth."

"I remember some such conundrum. Of course, it makes no sense to offer the eldest son eight and a half camels."

"Nor the youngest son one and eight ninths. But there is an ingenious solution to the problem."

"Yes, I remember. A wise man brings an extra camel of his own, raising the total to 18. The eldest son takes half of that number, namely, nine camels; the second son takes one third, or six camels; and the youngest son takes one ninth, or two camels. Those numbers total 17, whereupon the wise man departs once more with his own camel, and all are satisfied."

"Or at least, everybody thinks so. The psychology of the puzzle is almost as fascinating as the mathematics."

"But, Mustapha, you have more than 17 camels."

"Indeed, Allah has blessed me with 39. Moreover, I promised my father on his deathbed never to sell a camel. So it is not possible to reduce the number to 17. Of course, it would not be difficult to purchase a few extra camels should that prove necessary. The question I am unable to answer is whether some other collection of numbers would permit a similarly curious course of events."

"You could always triple everything," Ali said. "Start with 51 camels and the same disposition into fractions."

Mustapha nodded again, grimacing with pain. "I have thought of that, Ali. But then it would be necessary for the wise man to introduce three extra camels. That lacks elegance."

Ali rubbed his beard. "So the question is, what other numbers of camels would behave in this curious manner?"

"Yes. I had in mind assigning to each son some appropriate fraction of the total that would permit the introduction, and subsequent removal, of just one extra camel."

Ali leaned back and smiled. "Numbers, Mustapha, were always a strong point of mine. I wonder—" He gazed into space for a few seconds. "By the grace of Allah, there may be a way. But

THREE SONS share ownership of seven camels. The first son claims half the herd; the second son wants a quarter; the third demands an eighth. Can they each take their share without chopping up a camel?

first we must understand how the original trick works."

Mustapha scratched his head. "I confess I am sorely puzzled. The crucial camel appears and vanishes like a jinni from a lamp with a defective wick."

"It must be some quirk of the particular fractions chosen," Ali said. "For example, had there been 12 camels, with the sons getting one half, one third and one sixth, then the eldest would get six camels, the second four and the third two. No extra camel would be needed.... Aha! I believe I see a ray of light. The three fractions cannot possibly add up to one. If they did, such a trick would never work—for all camels would be divided with none left over. Let me see. What is the sum of $1/2$, $1/3$ and $1/9$?

"Ah, $17/18$," Mustapha said. "Of course! The sons inherit only $17/18$ of the total number of camels. If the total is 17, they can't divide the herd evenly. But if the total is 18, each son takes a portion of the 18 with one camel left over." A thought suddenly entered his mind. "He wasn't really a wise man, was he? He never pointed out to anyone that the fractions don't add up."

"In that omission lay his deepest wisdom," Ali countered. "The trick works because the sum of the three fractions assigned to the sons is a fraction whose denominator exceeds its numerator by one," Ali said. "Here the numerator is 17 and the denominator 18." He grinned broadly. "There are many such fractions. For any whole number d, they take the form $(d-1)/d$.... I've got it! You have 39 camels. Right?"

"Yes."

"Then all we need to do is choose fractions that sum to $39/40$," Ali said. "For example, $1/2$, $1/4$ and $9/40$." He turned in triumph, but then his face fell. "You seem unimpressed, Mustapha."

"It lacks simplicity, Ali. Each fraction should be one out of something. One out of three or one out of 19. Like that. Not nine out of 40."

"Ah. You require numerators of 1."

"Precisely."

"In short, you need a solution in whole numbers to the equation $1/a + 1/b + 1/c = (d-1)/d$. That is, the number $(d-1)/d$ must be expressed as a sum of three reciprocals. Egyptians often wrote down fractions in terms of a sum of reciprocals. Hence, the sum of $1/a$, $1/b$ and $1/c$ is known as a three-term Egyptian fraction."

"I found a way to simplify your equation," the patriarch said. He wrote down the following equation:

$$1/a + 1/b + 1/c + 1/d = 1.$$

Ali smote his thigh in delight. "So if a is 2, b is 3 and c is 9, then d must be 18 since $1/2 + 1/3 + 1/9 + 1/18 = 1$. And now all we have to do is find some other solutions to your four-term Egyptian equation. That is, find four numbers whose reciprocals sum to 1." He paused. "A reciprocal agreement, so to speak."

Mustapha's brow furrowed. "I can certainly think of one other solution," he said. "Namely, $1/4 + 1/4 + 1/4 + 1/4 = 1$. So what now?"

"We will find all possible solutions to your equation." Ali reached for a sheet of paper. "It is a delicate matter, for we are dealing with what mathematicians call a Diophantine equation, one that must be solved using only whole numbers. Indeed, in this case, positive whole numbers. Such equations were discussed by Diophantus of Alexandria around the third century."

Mustapha turned himself awkwardly in his bed to ease his shattered bones. "Are you not being overambitious, Ali, to seek all solutions? There could be a large number of them?"

"Diophantine equations tend not to have very many solutions," Ali replied, "although there are exceptions. And in this case—"

He began scribbling on the paper. "I believe we can prove that only a finite number of solutions exist. Moreover, the proof allows us to find them all in a systematic manner. Among them may be one that suits you. Suppose that the numbers are arranged in order of size, so $a \le b$ (a is less than or equal to b) and $b \le c \le d$. Then a must be at most 4. If a were to equal 5 or more, then b, c and d would equal five or more, and therefore the sum of their reciprocals would never equal 1 and would always be less than or equal to $1/5 + 1/5 + 1/5 + 1/5$, or $4/5$."

Mustapha stared at him. "This helps?"

"Yes. You see, we also know that all four numbers must be at least 2. Otherwise the sum begins $1/1$ and will always be too large. Therefore, only three cases need be considered: a equals 2, 3 or 4. In the first case, where $a = 2$, the equation becomes $1/2 + 1/b + 1/c + 1/d = 1$." He simplified this equation a bit and then wrote down the three cases. When $a = 2$, the sum of the reciprocals b, c and d equals $1/2$, that is $1/b + 1/c + 1/d = 1/2$. When $a = 3$, the sum must be equal to $2/3$, and when $a = 4$, the sum is equal to $3/4$.

Mustapha looked puzzled. "But, Ali, all you have done is replace one equation with three."

"Yes, Mustapha—but now each has only three variables instead of four! Moreover, I can repeat the same trick on each. For example, consider the first of the three equations, $1/b + 1/c + 1/d = 1/2$. It is apparent that the second smallest number, b, must be no more than 6. Otherwise, the sum would be less than or equal to $1/7 + 1/7 + 1/7$, or $3/7$, which is less than $1/2$. In the same manner, for three reciprocals that add up to $2/3$, b must be at most 4; and for a sum of $3/4$, b must also be at most 4. Thus, each of the three cases for the number a breaks up

into a finite number of subcases for *b*."

"And then," Mustapha said in excitement, "you use the same trick again!"

"Precisely. As I have said, if $1/b + 1/c + 1/d = 1/2$, then *b* must be at most 6. And since *a* is 2 in this case, *b* must be at least 3. Suppose, for example, that *b* is 3. Then $1/2 + 1/3 + 1/c + 1/d = 1$. That is, $1/c + 1/d = 1/6$."

"From which," Mustapha cried, "we deduce that *c* is at most 12, since $1/13 + 1/13$ equals $2/13$, which is less than $1/6$."

"Exactly. And that gives only a finite set of sub-subcases for *c*, after which *d* has a unique value that we can calculate precisely. For example, if $a = 2$, $b = 3$ and $c = 11$, then *d* must satisfy $1/2 + 1/3 + 1/11 + 1/d = 1$, which implies that *d* is $66/5$. But that is not a whole number, so there is no solution with $a = 2$, $b = 3$ and $c = 11$. On the other hand, if $a = 2$, $b = 3$ and $c = 10$, then $1/2 + 1/3 + 1/10 + 1/d = 1$, which implies that $d = 15$. This time a solution appears. In general, if *d* turns out to be a whole number, then we have found a solution; if not, then that particular sub-subcase does not lead to any solution.

"Moreover, the same argument applies to any equation of the form $1/a + 1/b + \ldots + 1/z = p/q$, where *a*, *b*, ..., *z*, *p* and *q* are positive whole numbers. There are only a finite number of ways to write any given fraction as an Egyptian fraction with a fixed number of terms. The solutions can be found by a series of simple deductions."

Mustapha coughed, then spat blood. "You seem to have proved a very general theorem, Ali."

"Precisely. Now, allow me a few moments while I calculate all possible solutions to your equation." Ali scribbled away furiously [*see table at left*]. "I find exactly 14 different solutions."

"And now the manner of your bequest stares us in the face," Ali pointed out. "The very first solution in the table is $1/2 + 1/3 + 1/7 + 1/42 = 1$. Mustapha, if you owned a total of 41 camels, you could decree that your eldest son should inherit one half of the herd, your second son should receive one third and your third son should get one seventh. Then if you die, Allah forbid, they will need to find a 42nd camel to satisfy your wishes. Then the eldest son will have 21 camels, the second son 14, and the third son six."

The dying man clasped the barber's hand. "Ali, you have answered my prayers. It merely remains for me to procure two more camels. Have the terms of the bequest drawn up immediately—"

There was a commotion outside the tent. Suddenly a small boy shot in through the flap. "Yes, Hamid? Do you normally approach the head of your family in such a precipitate fashion?"

"I apologize, master Mustapha ibn Mokhta. Your third wife, Fatima, has just borne you a son! Your fourth son!"

a	$1-\frac{1}{a}$	*b*	$1-\frac{1}{a}-\frac{1}{b}$	*c*	$1-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}$	*d*
2	1/2	2	0		0	
	1/2	3	1/6	6	0	
			1/6	7	1/42	42
			1/6	8	1/24	24
			1/6	9	1/18	18
			1/6	10	1/15	15
			1/6	11	5/66	
			1/6	12	1/12	12
	1/2	4	1/4	4	0	
			1/4	5	1/20	20
			1/4	6	1/12	12
			1/4	7	3/28	
			1/4	8	1/8	8
	1/2	5	3/10	5	1/10	10
			3/10	6	2/15	
			3/10	7	11/70	
	1/2	6	1/3	6	1/6	6
3	2/3	3	1/3	3	0	
			1/3	4	1/12	12
			1/3	5	2/15	
			1/3	6	1/6	6
	2/3	4	5/12	4	1/6	6
4	3/4	4	1/2	4	1/4	4

MUSTAPHA'S EQUATION—$1/a + 1/b + 1/c + 1/d = 1$—*has exactly 14 solutions if* a, b, c *and* d *are positive whole numbers and if* $a \le b \le c \le d$. *The colored boxes indicate cases in which* a, b, c *or* d *must equal a fraction or zero.*

FURTHER READING

RIDDLES IN MATHEMATICS: A BOOK OF PARADOXES. Eugene P. Northrop. Krieger, 1975.

UNSOLVED PROBLEMS IN NUMBER THEORY. H. Croft and R. K. Guy. Springer-Verlag, 1981.

```
Function Egyp(),
    Count: Ø,   A: 2,

Loop, T: 1-1/A, B: A,
    Loop, S: T-1/B,
        Block When S=Ø, Exit, C:B,
            Loop, R: S-1/C,
                Block When R=Ø, Exit, D:C,
                    Loop, Q= R-1/D,
                        Block When Q=Ø, Exit, E:D,
                            Loop, P: Q-1/E,
                                Block When P=Ø, Exit, F:E,
                                    Loop,
    Block When Recip (P-1/F),
        Count: Count +1, Exit, End Block,
                                F:F+1, When F>2/P, Exit,
                                    EndLoop, EndBlock
                        E:E+1, When E>3/Q, Exit,
                            End Loop, End Block,
                    D:D+1, When D>4/R, Exit,
                        EndLoop, End Block,
            C:C+1, When C>5/S, Exit,
                EndLoop, End Block,
        B:B+1, When B>6/T, Exit,
            End Loop,
A:A+1, When A>7, Exit,
    End Loop,
Count, EndFun;
```

$1-\frac{1}{A}-\frac{1}{B}-\frac{1}{C}-\frac{1}{D}-\frac{1}{E}-\frac{1}{F}-\frac{1}{G}=\emptyset$ A, B, C, D, E, F & G > ∅

```
1   rem to count the # of Egyptian fractions
10  Count = ∅
20  For A = 2 to 7
30    T = 1 - 1/A
40    For B = A to 6/T
50      S = T - 1/B
60      If S = ∅ then 240
70      For C = B to 5/S
80        R = S - 1/C
90        If R = ∅ then 230
100       For D = C to 4/R
110         Q = R - 1/D
120         If Q = ∅ then 220
130         For E = D to 3/Q
140           P = Q - 1/E
150           If P = ∅ then 210
160           For F = E to 2/P
170             O = P - 1/F      rem = 1/G
180             If O = ∅ then 200
190   If FP(1/O) = ∅ then Count = Count + 1
200           Next F
210         Next E
220       Next D
230     Next C
240   Next B
250 Next A
260 Print Count
```