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ON PARTITIONS OF BIPARTITE NUMBERS

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- 1. In statistical mechanics, we usually deal with assemblies containing a fixed number of particles in which energy is the only conserved quantity. Recently, Fermi (1) has shown that the angular distribution of the pions produced in high-energy nuclear collisions can be explained if one takes into account the conservation of angular momentum in addition to the conservation of energy. We are, therefore, led to discuss the thermodynamical properties of assemblies characterized by the conservation of two or more parameters. The simplest assumption of this type that we can make is that there are two parameters, say E and P, which are conserved, and that each particle of the assembly can occupy the levels (r,s) (r,s) are non-negative integers), where the contribution of the level (r,s) to E is re_0 and to P is $s\eta_0$. In order to find the entropy, and hence other thermodynamical properties of the system, we have to enumerate the distinct number of ways, p(m,n), in which an assembly of particles corresponding to given values of $E=mc_0$ and $P=n\eta_0$ can be realized. In this paper we find asymptotic expressions for p(m, n) in the following cases: (a) m is a fixed number, (b) m and n are of the same order. It is assumed here that the number of particles is greater than m and n. We deal with the case (a) in §2 and the case (b) in §4. §3 deals with the asymptotic expansions of the generating function for p(m, n) which are used in §4.
- 2. The partition function p(m,n) defined in §1 has also been studied previously by MacMahon (5) as the number of partitions of the 'bipartite' number $(m,n)^*$. We give in the following table the values of p(m,n) for m,n up to 5.

			$Table\ footnote{}$	Table for $p(m, n)$				
n	0	1	2	3	4		5	
m	1	1	2	3	5		7	. 7
1	21	2	4	7	12		19	70'
2	置と	4	9	16	29		47 —	201
3	13	7	16	31	57		97 —	201
4	H 2	12	29	57	109		189	412
5	67	19	47	97	189		338	11/16
ı illustr	ation	ther	line partitions of t	the hipartit	e number 19	9) an	o given hal	-49,465

As an illustration the nine partitions of the bipartite number (2, 2) are given below: (2, 2); (2, 0), (0, 2); (1, 1), (1, 1); (2, 1), (0, 1); (1, 2), (1, 0); (2, 0), (0, 1), (0, 1); (0, 2), (1, 0), (1, 0), (1, 0), (1, 0), (1, 0), (1, 0), (1, 0), (0, 1), (0, 1).

^{*} It may be noted that any bipartite partition of (m, n) can be represented by a matrix (a_{rs}) with (m+1) rows and (n+1) columns, where a_{rs} is the number of times the summand (r, s) occurs in the partition such that $\sum_{r=0}^{m} \sum_{s=0}^{n} ra_{rs} = m$, and $\sum_{r=0}^{m} \sum_{s=0}^{n} sa_{rs} = n$.

If one of the integers m is fixed, we can express p(m,n) in terms of the partition functions p(r) of unipartite numbers. The generating function of p(m,n) is

$$Z(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p(m,n) x^m y^n$$

$$= \frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2) \dots}$$

$$= \prod_{r=1}^{\infty} (1-x^r)^{-1} \prod_{s=1}^{\infty} \prod_{t=0}^{\infty} (1-y^s x^t)^{-1}.$$
(1)

The product converges for |x| < 1, |y| < 1. Since (MacMahon (6))

$$\frac{1}{(1-a)(1-ax)(1-ax^2)(1-ax^3)\dots} = 1 + \frac{a}{1-x} + \frac{a^2}{(1-x)(1-x^2)} + \frac{a^3}{(1-x)(1-x^2)(1-x^3)} + \dots,$$

we can write the generating function Z in the form

$$Z(x,y) = \prod_{r=1}^{\infty} (1-x^r)^{-1} \prod_{s=1}^{\infty} \left(\sum_{t=0}^{\infty} a_t y^{st} \right)$$
$$= \prod_{r=1}^{\infty} (1-x^r)^{-1} (1+b_1 y + b_2 y^2 + \dots), \tag{2}$$

where

$$a_0 = 1$$
, $a_1 = \frac{1}{1-x}$, $a_2 = \frac{1}{(1-x)(1-x^2)}$, $a_3 = \frac{1}{(1-x)(1-x^2)(1-x^3)}$, ...
$$b_m = \sum_{\sum x_1 = -m} a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_s} \quad (m = 0, 1, 2, \dots).$$

and

 a_{λ_r} is the coefficient of y^{rt} from the series $\sum_{t=0}^{\infty} a_t y^{rt}$. Now there are p(m) terms in b_m , and

the only term in b_m which has the maximum number of m factors is

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^m)},$$

and therefore for |x| < 1 and $m \ge 3$

$$\left| b_m - \frac{1}{(1-x)(1-x^2)\dots(1-x^m)} \right| < \frac{p(m)}{(1-|x|)^{m-1}}.$$

It follows that for $x = e^{-\lambda}$, $\lambda = \sigma + it$

$$b_m = \frac{1}{m! \lambda^m} \{ 1 + O(|\lambda|) \}, \tag{3}$$

where $\lambda \to 0$ in the Stolz angle $|t| \le \Delta \sigma$ $(0 < \Delta < \infty)$, provided m is a fixed positive integer. In this case $b_m \prod_{r=1}^{\infty} (1 - x^r)^{-1} \sim \frac{\exp(\pi^2/6\lambda)}{m! (2\pi)^{\frac{1}{2}} \lambda^{m-\frac{1}{2}}}.$ (4)

Since p(m, n) is the coefficient of x^n in the expansion of $b_m \prod_{r=1}^{\infty} (1-x^r)^{-1}$, we can apply

Ingham's Tauberian Theorem (4) to (4), if we can prove that p(m,n) is an increasing function of n for a fixed m. Now from (2) we have identically

$$p(m,n) = \sum_{r=0}^{n} c_{m,r} p(n-r),$$
 (5)

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ı_{rs}) urs where the coefficients $c_{m,r}$ are positive functions of m and r only. It follows that p(m,n), for a fixed m, is an increasing function of n. Thus we obtain finally for $n \to \infty$ and a fixed m

 $p(m,n) \sim \left(\frac{\sqrt{(6n)}}{\pi}\right)^m \frac{1}{(4\sqrt{3})n} \exp\left[\pi \left(\frac{2n}{3}\right)^{\frac{1}{2}}\right]. \tag{6}$

3. We now obtain an asymptotic expansion for the generating function Z(x,y). Substituting $x = e^{-\lambda}$, $y = e^{-\mu}$ in Z(x,y), we obtain for $\Re(\lambda) > 0$, $\Re(\mu) > 0$,

$$\log Z = -\sum_{\substack{r=0 \ s=0 \ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \log \left(1 - e^{-r\lambda - s\mu}\right) = \sum_{\substack{r=0 \ s=0 \ r+s>0}}^{\infty} \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \frac{1}{t} e^{-t(r\lambda + s\mu)} = \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \frac{e^{-ts\mu}}{t} + \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} \frac{1}{t} \frac{e^{-tr\lambda}}{1 - e^{-t\mu}}$$

$$= \sum_{t=1}^{\infty} \frac{1}{t} \frac{1}{e^{t\mu} - 1} + \sum_{t=1}^{\infty} \frac{e^{t\mu}}{t(e^{t\lambda} - 1)(e^{t\mu} - 1)} = \sum_{t=1}^{\infty} \frac{1}{t} \left\{ \frac{1}{e^{t\lambda} - 1} + \frac{1}{e^{t\mu} - 1} + \frac{1}{(e^{t\lambda} - 1)(e^{t\mu} - 1)} \right\}$$

$$= \frac{1}{\lambda} \sum_{t=1}^{\infty} \frac{1}{t^2} \frac{1}{e^{t\mu} - 1} + \frac{1}{\mu} \sum_{t=1}^{\infty} \frac{1}{t^2} \frac{1}{e^{t\lambda} - 1} - \frac{\zeta(3)}{\lambda \mu} + \frac{3}{4} \sum_{t=1}^{\infty} \frac{1}{t} \left\{ \frac{1}{e^{t\lambda} - 1} + \frac{1}{e^{t\mu} - 1} \right\}$$

$$+ \frac{\zeta(2)}{4} \left(\frac{1}{\lambda} + \frac{1}{\mu} \right) + \sum_{t=1}^{\infty} \frac{1}{t} \left\{ \left(\frac{1}{e^{t\lambda} - 1} - \frac{1}{t\lambda} \right) \left(\frac{1}{e^{t\mu} - 1} - \frac{1}{t\mu} \right) + \frac{1}{4} \left(\frac{1}{e^{t\lambda} - 1} - \frac{1}{t\mu} \right) \right\}. \tag{7}$$

If λ and μ lie in the Stolz angles

$$G_1: \quad |\Im(\lambda)| \leq \Delta \Re(\lambda) \quad (0 < \Delta < \infty),$$

$$G_2: \quad |\Im(\mu)| \leq \Delta \Re(\mu) \quad (0 < \Delta < \infty),$$

the series in (7) for $\lambda\mu \log Z$ are uniformly convergent in any bounded portions of G_1 and G_2 , so that we can take limits term by term and therefore we get the asymptotic expression $\log Z \sim \zeta(3)/\lambda\mu, \tag{8}$

when $\lambda \to 0$ and $\mu \to 0$ in G_1 and G_2 .

We will obtain an asymptotic expansion for $\log Z$ for real positive values of λ and μ , which is all that we require in §4 for evaluating p(m,n).

Let
$$u = \sum_{t=1}^{\infty} \frac{1}{t^2} \frac{1}{e^{l\mu} - 1}.$$
 Then
$$\frac{du}{d\mu} = -\sum_{t=1}^{\infty} \frac{1}{t} \frac{e^{t\mu}}{(e^{l\mu} - 1)^2}$$
$$= -\sum_{t=1}^{\infty} \sum_{r=1}^{\infty} \frac{r}{t} e^{-rl\mu}$$
$$= \sum_{t=1}^{\infty} r \log (1 - e^{-r\mu}).$$

Using a result of Wright (7) we obtain

$$\frac{du}{d\mu} = -\frac{\zeta(3)}{\mu^2} - \frac{1}{12} \log \mu - \zeta'(-1) + O(\mu^2),$$
 and therefore
$$u = \frac{\zeta(3)}{\mu} - \frac{1}{12} \mu (\log \mu - 1) - \mu \zeta'(-1) + A_1 + O(\mu^2),$$
 (9)