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An Ordering of the Rationals

551. [November, 1964] *Proposed by Herta Taussig Freitag, Hollins College, Virginia.*

According to Cantor's "diagonal procedure," the denumerability of the rationals may be established by ordering them in the manner indicated below:

$$
\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdot & \cdot & \cdot \\[1ex]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdot & \cdot & \cdot \\[1ex]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdot & \cdot & \cdot \\[1ex]
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Thus,

$$\frac{1}{1} \leftrightarrow 1, \qquad \frac{1}{2} \leftrightarrow 2, \qquad \frac{2}{1} \leftrightarrow 3, \text{ etc.}$$

Design a matching formula between any given fraction a/b and the corresponding natural number n.

Solution by Henry W. Gould, West Virginia University.

There are two parts to this problem: (i) To show that to any given rational p/q we may assign a unique natural number n; (ii) To show that for a given natural number n we may exhibit a unique rational p/q. We give a constructive solution of both parts.

(i) Evidently a formula for n depends on the parity of $p+q$, this being so because of the alternating way in which the diagonals are traced out. We have in fact

$$n = \frac{(p+q-1)(p+q-2)}{2} + q, \quad \text{if } p+q \text{ is even}$$

and

$$n = \frac{(p+q-1)(p+q-2)}{2} + p, \quad \text{if } p+q \text{ is odd.}$$

These are based on nothing more complicated than the observation that n is given by counting in unit steps from one triangular number (1, 3, 6, 10, 15, 21, $\cdots$) to the next.

(ii) To find a constructive way of actually exhibiting which rational p/q is assigned to a given natural number n we make use of the formula

$$(*) \qquad a = a_n = \left[\frac{1 + [\sqrt{(8n-7)}]}{2}\right],$$

which, for natural numbers n, generates the curious sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \cdots,$$

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each natural number k occurring precisely k times here. The formula in this form was called to the author's attention by some lecture notes of Leo Moser (Canadian Mathematical Congress lectures). In the formula, $[x]$ denotes the greatest integer $\leqq x$. Essentially the same formula occurs in E. S. Keeping's solution to Problem E1164 in the *American Mathematical Monthly* (1955, p. 731), another problem about finding which rational is assigned in a certain ordering. The solution of another *Monthly* problem, E1382, can also be made to depend on the same formula. An older reference of interest is to Problem 91, Page 271, in the 1939 edition (1953 reprint) of *Advanced Algebra* by S. Barnard and J. M. Child.

In any event, it is easy to determine from the formula that the sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \cdots$ is generated by $n - \binom{a}{2}$, and the sequence is generated by

$$\binom{a+1}{2} - (n-1).$$

These are the two sequences on which the pattern is based, and so we easily find that the particular p and q corresponding to a given value of n may be determined as follows:

We always have $p+q=a+1$, a being given by (*). Then

$$\begin{aligned} n - \binom{a}{2} &= p, \quad \text{if } p+q \text{ is odd,} \\ &= q, \quad \text{if } p+q \text{ is even;} \\ \binom{a+1}{2} - (n-1) &= q, \quad \text{if } p+q \text{ is odd,} \\ &= p, \quad \text{if } p+q \text{ is even.} \end{aligned}$$

An example will illustrate the ease of calculation. What is the 1000th rational assigned in the ordering? We have

$$p+q = a+1 = 1 + \left[\frac{1+[\sqrt{(7993)}]}{2}\right] = 46.$$

Then

$$p = \binom{46}{2} - 999 = 36 \quad \text{and} \quad q = 1000 - \binom{45}{2} = 10,$$

so that the 1000th rational is 36/10. This is easily checked by part (i); indeed $\frac{1}{2}(45)(44)+10=1000$.

Also solved by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; John A. Burslem, St. Louis University; Alan K. Chelgren, Centre College, Kentucky; Robert E. Cohen, University of Chicago; R. J. Cormier, Northern Illinois University; Ronald DeLaite, Orono, Maine; Monte Dernham, San Francisco, California; Robert V. Esperti, General Motors Defense Research Laboratories; Philip Fung, Fenn College, Ohio; Edwin V. Gadecki, Technical Operations Research, Burlington, Massachusetts; Harry W. Hickey, Arlington, Virginia; Roy H. Hines, Concord, Massachusetts; Stephen Hoffman, Trinity College, Connecticut; Richard A. Jacobson, South Dakota State University; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Herbert R. Leifer, Pittsburgh, Pennsylvania; Douglas Lind, University of Virginia; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Wade E. Philpott, Lima, Ohio; B. E. Rhoades, CUPM, Berkeley, California; Lawrence A. Ringenberg, Eastern Illinois University; J. R. Senft, DePaul University Chicago; Arnold Singer, Institute of Naval Studies, Cambridge, Massachusetts; Sidney Spital, California State Polytechnic College; Myron Tepper, University of Illinois; A. M. Vaidya, Texas Technological College; James A. Will, SUNY at Fredonia, New York; Charles Ziegenfus, Madison College, Virginia; and the proposer.

Rhoades found a solution in Zehna and Johnson, *Elements of Set Theory*, Boston, 1962, p. 108.

Space does not permit the publication of a number of interesting and different ways of matching p/q and n that were submitted.

Comment on Q343

Q343 [September, 1964] *Comment by Charles W. Trigg, San Diego, California.*

The published answer states "The angle θ evidently is 15°." It is further evident that θ is $15° \pm k\cdot 360°$, and for the right and left hand expressions, θ also is $135° \pm k\cdot 360°$.

Comment on Q347

Q347 [September, 1964] *Comment by Charles W. Trigg, San Diego, California.*

The method given shows

$$(1234)^2 = (1000)^2 + (200)^2 + (30)^2 + 4^2 + 2(1000)200 + 2(1200)30 + 2(1230)4.$$

This involves *thirteen* digit multiplications and the writing down of *seventeen* digits (exclusive of the final product):

```
1 0 0 0 0 0 0
    4   9 1 6
  4
    7 2
      9 8 4
-------------
1 5 2 2 7 5 6
```

The zeros are necessary as place locaters.