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NOTE ON A NONLINEAR RECURRENCE RELATED TO $\sqrt{2}$

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In a recent study of sorting algorithms for a partially-sorted set, F. K. Hwang and S. Lin [2] introduced the following sequence:

$$a_1 = 1, \qquad a_{n+1} = \left[\sqrt{2a_n(a_n+1)}\right], \qquad n \geqq 1,$$

where $[\cdot]$ denotes the greatest integer function. Thus, the sequence begins

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
a_n	1	2	3	4	6	9	13	19	27	38	54	77	109	

1521

One notes that $a_{2n+1} - a_{2n} = 2^{n-1}$ for $1 \leqq n \leqq 6$ and it might be conjectured that this holds in general.

In this note we investigate the sequence $\{a_n\}$. We obtain an explicit expression for a_n from which the conjecture follows as well as the following curious result: $a_{2n+1} - 2a_{2n-1}$ is just the nth digit in the binary expansion of $\sqrt{2}$.

We begin by making the preliminary observation that if $S(\alpha)$ denotes the set of integers $\{[\alpha], [2\alpha], [3\alpha], \cdots\}$ then every positive integer occurs in exactly one of the two sets $S(1+1/\sqrt{2})$ and $S(1+\sqrt{2})$. This follows from well-known results for $S(\alpha)$ (cf. [1]) together with the fact that $(1+1/\sqrt{2})^{-1} + (1+\sqrt{2})^{-1} = 1$ and $1+\sqrt{2}$ is irrational. Thus, for any positive integer m there exists a unique integer t such that either $m = [t(1+1/\sqrt{2})]$ or $m = [t(1+\sqrt{2})]$.

THEOREM. *Let* $a_1 = m$, $a_{n+1} = [\sqrt{2a_n(a_n+1)}]$, $n \geqq 1$. *Then*

$$a_n = \begin{cases} [t(2^{(n-1)/2} + 2^{(n-2)/2})] & \text{if } m = \left[t\left(1+\dfrac{1}{\sqrt{2}}\right)\right] \\ [t(2^{n/2} + 2^{(n-1)/2})] & \text{if } m = [t(1+\sqrt{2})]. \end{cases}$$

Proof. First note that since no integral square lies between

$$2t^2 + 2t \quad \text{and} \quad 2t^2 + 2t + \tfrac{1}{2} = 2(t+\tfrac{1}{2})^2$$

then

$$[\sqrt{2a_n(a_n+1)}] = [\sqrt{2}(a_n+\tfrac{1}{2})].$$

Thus we can assume

$$a_{n+1} = [\sqrt{2}(a_n+\tfrac{1}{2})], \qquad n \geqq 1.$$

Suppose $x=t(1+1/\sqrt{2})$ for some positive integer t. Then

$$[\sqrt{2}([x]+\tfrac{1}{2})] = [\sqrt{2}\,x] \tag{1}$$

Proof of (1). Let

$$\beta = \frac{t}{\sqrt{2}} - \left[\frac{t}{\sqrt{2}}\right].$$

Then $x=[x]+\beta$. Also

$$\sqrt{2}\,x = t(1+\sqrt{2}), \qquad \sqrt{2}\,t - [\sqrt{2}\,t] \equiv \beta'$$

and

$$2\beta = \beta' + \alpha_1, \qquad \alpha_1 = 0 \quad \text{or } 1,$$

(i.e., $\beta=\cdot\alpha_1\alpha_2\cdots$ $\beta'=\cdot\alpha_2\alpha_3\cdots$ expressed base 2).

$$\therefore\ (1) \text{ iff } [t(\sqrt{2}+1)]\left[=\sqrt{2}\left(t\left(1+\frac{1}{\sqrt{2}}\right)-\beta+\tfrac{1}{2}\right)\right]$$

$$\text{iff } t+[t\sqrt{2}] = \left[t+[t\sqrt{2}]+\beta'-\sqrt{2}\,\beta+\frac{1}{\sqrt{2}}\right]$$

$$\text{iff } 0 = \left[\beta'-\sqrt{2}\,\beta+\frac{1}{\sqrt{2}}\right] = \left[\beta'-\frac{\sqrt{2}}{2}(\beta'+\alpha_1)+\frac{1}{\sqrt{2}}\right]$$

$$\text{iff } 0 = \left[\beta'\left(1-\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}(1-\alpha_1)\right].$$

The expression inside is $\geqq 0$. Also $\beta'<1$ so that the expression inside is $<1-1/\sqrt{2}+1/\sqrt{2}\cdot 1=1$. $\therefore$ (1) holds.

Next, suppose $x=t(1+\sqrt{2})$, for some positive integer t Then

$$[\sqrt{2}([x]+\tfrac{1}{2})] = [\sqrt{2}\,x]. \tag{1'}$$

Proof of (1′). As before, let $\beta'=\sqrt{2}\,t-[x\sqrt{2}]$. Then $x\sqrt{2}=t(2+\sqrt{2})$ and $x\sqrt{2}-[x\sqrt{2}]=t\sqrt{2}-[t\sqrt{2}]=\beta'$.

$$\therefore\ (1') \text{ iff } [2t+t\sqrt{2}] = \left[\sqrt{2}\,t(1+\sqrt{2})-\sqrt{2}\,\beta+\frac{1}{\sqrt{2}}\right]$$

$$\text{iff } 2t+[t\sqrt{2}] = \left[2t+[\sqrt{2}\,t]+\beta-\sqrt{2}\,\beta+\frac{1}{\sqrt{2}}\right]$$

$$\text{iff } 0 = \left[\beta(1-\sqrt{2})+\frac{1}{\sqrt{2}}\right].$$

But

$$0 < \frac{2-\sqrt{}}{2} = 1 - \sqrt{2} + \frac{1}{\sqrt{2}} < \beta(1-\sqrt{2}) + \frac{1}{\sqrt{2}} \leqq \frac{1}{\sqrt{2}} < 1;$$

$\therefore$ (1′) holds. Hence,

$$\text{if} \quad m = \left[t\left(1 + \frac{1}{\sqrt{2}}\right)\right] = a_n \quad \text{then} \quad a_{n+1} = [t(\sqrt{2}+1)];$$

$$\text{if} \quad m = [t(\sqrt{2}+1)] = a_n \quad \text{then} \quad a_{n+1} = \left[2t\left(1 + \frac{1}{\sqrt{2}}\right)\right].$$

A minor induction argument on n now proves the theorem.

We point out that it is possible to express the conclusion of the theorem in a somewhat more concise form:

If $a_1 = m$ and $a_{n+1} = [\sqrt{2a_n(a_n+1)}]$ then

$$a_n = [\tau(2^{(n-1)/2} + 2^{(n-2)/2})], \qquad n > 1,$$

where τ is the mth smallest real number in the set $\{1, 2, 3. \cdots\} \cup \{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \cdots\}$. The first few values are:

m	1	2	3	4	5	6	7	8	9	10	11	12	
τ	1	$\sqrt{2}$	2	$2\sqrt{2}$	3	4	$3\sqrt{2}$	5	$4\sqrt{2}$	6	7	$5\sqrt{2}$	$\cdots$.

The fact that for $m=1$, $a_{2n+1} - 2a_{2n-1}$ is the nth digit in the binary expansion of $\sqrt{2}$ is now immediate (as is the fact $a_{2n+1} - a_{2n} = 2^{n-1}$).

It would be interesting to know if similar results hold for sequences defined by

$$a_{n+1} = [\sqrt{3a_n(a_n+1)}], \qquad a_{n+1} = [\sqrt[3]{2a_n(a_n+1)(a_n+2)}], \text{ etc.}$$

References

1. I. Niven, Diophantine Approximations, Wiley, New York, 1963.

2. F. K. Hwang and S. Lin, An Analysis of Ford and Johnson's Sorting Algorithm, to appear in Proc. 3rd Annual Princeton Conference on Information Sciences and Systems.