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The Number Of Non-Isomorphic m-Graphs[1)]

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Since several years the graph-theory is interested in generalized graphs. We start with a set $N=\{1,\dots,n\}$ of vertices. By a generalized edge of an m-graph we mean an unordered set $\{a_1,\dots,a_m\}$ of vertices. For m=2 we have the classical graphs (undirected, without loops).

We want to know the number $G_n^{(m)}$ of non-isomorphic m-graphs over N. We are interested in an exact formula, but even more in an asymptotic approximation for $n\to\infty$. Two m-graphs G_1, G_2 are isomorphic iff there is a permutation $\pi\in\mathfrak{S}_n$ which transforms the edges of G_1 into the edges of G_2. We want to prove the

Theorem:
For $m\geq 2$: $$G_n^{(m)} = \frac{2^{\binom{n}{m}}}{n!}\left(1 + \frac{\binom{n}{2}}{2^{\binom{n-2}{m-1}}}(1+o(1))\right).$$

Corollary[2)]: $$G_n^{(m)} \sim \frac{2^{\binom{n}{m}}}{n!}$$

Classical Case: m=2: $$G_n^{(2)} = \frac{2^{\binom{n}{2}}}{n!}\left(1 + \frac{2n^2-2n}{2^n}(1+o(1))\right)$$

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1) This paper was read at the Mathematical Institute Oberwolfach, Germany, on July 3,1967 at a meeting on graph the

2) The corollary was first announced for the classical case m=2 by HARARY [2], p. 258. The better approximation in the classical case appears in [3], §4, Satz 2^L.

We can interpret the result in terms of the order of the automorphism-group of m-graphs: If all those m-graphs would have the trivial automorphism group, we would have $G_n^{(m)}$ to be exactly $\frac{1}{n!}\cdot 2^{\binom{n}{m}}$. So our result says, that nearly all graphs have the trivial automorphism-group, and such cases like the complete m-graph or the null-graph, who admit the whole $\mathfrak{S}_n$, are exceptions.

We here only confine for convenience to the "pure" case. As variants of the problem one can solve in an analogous manner the following generalizations:

a) Admit all μ-sets with $1\leq\mu\leq m$ $(m\geq 2)$, so that we admit lower dimensional edges like loops etc.

b) Admit manyfold occurences of the same edges, which are distinguished by several colours.[3]

c) Take directed m-graphs (m-place relations) or even finite systems of various-placed relations over N. These problems are even easier to solve than those discussed here[4].

The proof of the theorem uses the famous counting theory of POLYA (1937). Without going into the details of this theory, we can state the most important formula (1) below with the following concepts: Every $\pi\in\mathfrak{S}_n$ induces a permutation $\Pi=\Pi(\pi)$ of a so-called m-Line-Group $\mathfrak{L}_n^{(m)}$, which is a permutation group over the $\binom{n}{m}$ m-sets over N of order n!. If π has the

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3) The cases a) and b) are handled with the method of Hadamard products mentioned in [4].

4) Cf. [4], Theorem 3.

cycle partition $\gamma(\pi) = (p_1, \ldots, p_n)$, which means, that π has p_j cycles of length j, then the induced Permutation Π has the partition $\eta(\Pi) = (P_1, \ldots, P_{\binom{n}{m}})$.

Now the Polya-theory gives us a straightforward proof for [5]

$$(1) \quad G_n^{(m)} = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} 2^{P_1 + \ldots + P_{\binom{n}{m}}}.$$

The problem is "only" the computation of the numbers P_j. This has been solved in the case m=2 by Polya and Harary and in a form, which is somewhat simpler, by the author[6]. We also have a formula for $m=3$[7], and it is laborious, though in principle possible to write down the formulas for each m. Fortunately for our purpose of asymptotic computation we only need the numbers P_1, as will be seen under (3). Here we have

$$(2) \quad P_1(\pi) = \sum_{q = (q_1, \ldots, q_m)} \binom{p_1}{q_1} \cdots \binom{p_m}{q_m},$$

where the sum is over all partitions of m.

Proof of (2): The m-set (edge) M is fixed by π iff the following is true: M contains all elements of a cycle of π, if it contains one element of this cycle at all. Given a partition $(q_1, \ldots, q_m)$ of m, we have the choice of p_j cycles of π for the q_j cycles needed in M.

Now our theorem is proved by using the idea, that the main part of the formula of the theorem is given by the identical permutation $\pi = \varepsilon$ in (1), the correction term is given by the $\binom{n}{2}$

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5) Cf. [3], §1, Satz 3^{L_m} for the two-dimensional case.
6) Cf. [1], p.451 (10) and [3], §2, Satz 1^{L_2}.
7) Cf. [3], §2, Satz 1^{L_3}.

transpositions τ of the $\mathcal{S}_n$, and all other permutations are contained in o(1).

I) $\pi = \varepsilon$. $\gamma(\pi) = (n,0,\ldots,0)$, so $\mathbb{P}(\Pi) = (\binom{n}{m},0,\ldots,0)$. The term in (1) is $\frac{1}{n!}\cdot 2^{\binom{n}{m}}$.

II) Let $\pi = \tau$ a transposition. Since $\gamma(\tau) = (n-2,1,0,\ldots,0)$, we have with (2) $P_1(\pi) = \binom{n-2}{m} + \binom{n-2}{m-2}$, and we see elementarily, that $P_2 = \binom{n-2}{m-1}$. All other P_j are zero. So we have as term in (1)

$$\frac{1}{n!}\binom{n}{2}\cdot 2^{\binom{n-2}{m}+\binom{n-2}{m-2}+\binom{n-2}{m-1}} = \frac{1}{n!}\binom{n}{2}\cdot 2^{\binom{n}{m}-\binom{n-2}{m-1}},$$

the second term of our theorem.

III) Now we have to look for all other permutations. Let $\gamma(\pi) = (n-\varkappa, p_2,\ldots,p_n)$ with $\varkappa \geq 3$. Let

$$\Sigma_{\varkappa,n} := \frac{1}{n!}\sum_{\pi \text{ with } p_1 = n-\varkappa} 2^{H(\pi)},$$

where $H(\pi)$ is an abbreviation for the exponent in (1). We state the following:

a) There are at most $2^{\varkappa}$ partitions $(p_1,\ldots,p_n)$ of n with $p_1 = n-\varkappa$.

b) The number of permutations to the partition $(p_1,\ldots,p_n)$ is known to be $$\frac{n!}{p_1!\ldots p_n!\;1^{p_1}\ldots n^{p_n}} \leq \frac{n!}{(n-\varkappa)!} \leq n^{\varkappa}.$$

c) $$H(\pi) = P_1 + \ldots + P_{\binom{n}{m}} \leq P_1 + \tfrac{1}{2}\left(2P_2 + 3P_3 + \ldots + \binom{n}{m}P_{\binom{n}{m}}\right)$$
$$= P_1 + \tfrac{1}{2}\left(\binom{n}{m} - P_1\right) = \tfrac{1}{2}\left(\binom{n}{m} + P_1\right).$$

So we only need P_1 in the estimation of $\Sigma_{\varkappa,n}$. Thus we have

(3) $\sum_{\varkappa,n} \le \frac{1}{n!}(2n)^{\varkappa}\cdot 2^{\frac{1}{2}(\binom{n}{m}+P_1)} = \frac{1}{n!}(2n)^{\varkappa}\frac{2^{\binom{n}{m}}}{2^{\frac{1}{2}(\binom{n}{m}-P_1)}} = \frac{(2n)^{\varkappa}\,2^{\binom{n}{m}}}{n!\,2^{\frac{1}{2}E}}$,

say. Now we evaluate P_1 from (2) for this fixed $\varkappa$:

$P_1 := \binom{n-\varkappa}{m} + \sum_{\lambda=2}^{m}\ \sum_{\substack{q\\ q_1=m-\lambda}} \binom{n-\varkappa}{q_1}\binom{p_2}{q_2}\cdots\binom{p_n}{q_n}$. Each term of the sum is

$$\le \binom{n-\varkappa}{m-\lambda}p_2^{q_2}\cdots p_n^{q_n} \le (n-\varkappa)^{m-\lambda}n^{q_2+\cdots+q_n} \le n^{m-\lambda+(q_2+\cdots+q_n)}$$

Since $q_2+\cdots+q_n \le \frac{1}{2}(2q_2+3q_3+\cdots+mq_m) = \frac{1}{2}(m-q_1) = \frac{\lambda}{2}$, we can continue $\le n^{m-\frac{\lambda}{2}}$. Since the sum has finitely many terms, and since the sums for $\lambda = 2,3$ turn out explicitly to be $\varkappa O(n^{m-2})$, we have

(4) $P_1 = \binom{n-\varkappa}{m} + \varkappa O(n^{m-2})$, the O-constant depending on m and not on $\varkappa$. Now we can compute the Exponent E in (3): We have by (4) $\binom{n}{m} - P_1 = \binom{n}{m} - \binom{n-\varkappa}{m} - \varkappa O(n^{m-2})$,

$= \frac{m\varkappa}{m!}n^{m-1}(1+O(\frac{1}{n})) - O(n^{m-2})$, the first O-constant depending on $\varkappa$, and finally

(5) $E = \frac{\varkappa}{(m-1)!}n^{m-1}(1+o(1))$, the o-constant depending on $\varkappa$

We also want to estimate the difference $\binom{n}{m} - \binom{n-\varkappa}{m} - O(n^{m-2})$ <u>uniformly</u> in $\varkappa$. Let $T(x) := \binom{x}{m}$. Using Taylor's formula we get

$T(n-\varkappa) = T(n) - \frac{\varkappa}{1!}T'(n) + \frac{\varkappa^2}{2!}T''(n) \mp \ldots$, or

$\binom{n}{m} - \binom{n-\varkappa}{m} = T(n) - T(n-\varkappa) = \varkappa T'(n)\left(1 - \frac{\varkappa}{2!}\cdot\frac{T''(n)}{T'(n)} + \frac{\varkappa^2}{3!}\cdot\frac{T'''(n)}{T'(n)} \mp \cdots\right.$

$= T'(n).B$, say. Now one can prove for the bracket

$B > \frac{1}{2m}$ for n sufficiently large [8]. For $n \geq n_0(m)$ the correction term $O(n^{m-2})$ does not destroy the estimation. So we have

(6) $\binom{n}{m} - P_1 \geq \varkappa T'(n) \cdot \frac{1}{2m}$ for $n \geq n_0(m)$.

Using this fact, we have by (3)

(7) $\sum_{\varkappa, n} \leq \frac{1}{n!} \cdot 2^{\binom{n}{m}} \cdot \left(\frac{2n}{2^{\frac{1}{2} T'(n) \cdot \frac{1}{2m}}} \right)^{\varkappa}$. Since $T'(n) \sim \frac{n^{m-1}}{(m-1)!}$

and $m > 1$, we have $T'(n)$ of larger order than zero, so that the bracket in (7) is $< \frac{1}{2}$ for $n \geq n_1(m)$. Now we split the remainder: $\sum_{\varkappa=3}^{n} \sum_{\varkappa, n} = \sum_{\varkappa=3}^{4m} \sum_{\varkappa, n} + \sum_{\varkappa=4m+1}^{n} \sum_{\varkappa, n}$.

a) The finitely many members of the first sum have by (3) and (5) the form

$$\sum_{\varkappa, n} \leq \frac{2^{\binom{n}{m}}}{n!} \cdot \frac{(2n)^{\varkappa}}{2^{\frac{1}{2} \cdot \frac{\varkappa}{(m-1)!} n^{m-1} (1 + o(1))}}$$

. Since $\varkappa > 2$, $\frac{\varkappa}{2} \cdot \frac{1}{(m-1)!} \cdot n^{m-1}$ is of larger order than $\binom{n-2}{m-1} \sim \frac{n^{m-1}}{(m-1)!}$ in the correction term of our theorem. The terms $(2n)^{\varkappa}$ are by far of smaller order than those we have just discussed.

b) The second sum is estimated for large n by the infinite geometrical series: According to (7) we have

$$\sum_{\varkappa=4m+1}^{n} \sum_{\varkappa, n} \leq \frac{1}{n!} \cdot 2^{\binom{n}{m}} \cdot 2 \cdot \frac{(2n)^{4m+1}}{2^{T'(n) + \frac{T'(n)}{m}}}.$$

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8) Cf. [4], Lemma to theorem 3.

Now $T'(n) \sim \frac{n^{m-1}}{(m-1)!}$ has just the order of the $\binom{n-2}{m-1}$ in the correction term, and the additional $\frac{T'(n)}{m}$ makes the remainder of effectively smaller order than the correction term.

References

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N o t e

On Structure Numbers For Finite Relational Systems

A problem of Carnap (1950) is to determine the number $S(n,\tau)$ of non-isomorphic relational systems $\mathcal{R}$ of type $\tau = [\mu_1, \ldots, \mu_m]$ over a finite set $N = \{1,\ldots,n\}$, where μ_i is the number of i-plac[e] relations in $\mathcal{R}$. Let $T(z) := \mu_m z^m + \ldots + \mu_1 z$. The type group $\mathcal{T}_n^\tau$ operates over $T(n)$ tuples and has a cycle index $Z(\mathcal{T}_n^\tau)$, which is a sort of Hadamard product of the cycle indices of the so called i-tuple-groups (cf. [2]). Using Polya's combinatorial theory, we get

Theorem 1:

$$S(n,\tau) = \frac{1}{n!} \sum_{\pi \in \mathcal{T}_n} 2^{\mu_m (P_1^{(m)} + \ldots + P_{n^m}^{(m)}) + \ldots + \mu_1 (P_1^{(1)} + \ldots + P_n^{(1)})}$$

Here $P_j^{(i)}$ is the number of cycles of length j for that permutation $\overline{\pi}$ of the i-tuple-group, which is induced by π.

The following asymptotic result shows, that nearly all relation[al] systems only have the trivial automorphism group (cf. [1]):

Theorem 2:

For $m \geq 2$, $$S(n,\tau) = \frac{2^{T(n)}}{n!} \left(1 + \binom{n}{2} \cdot 2^{\frac{1}{2}(T(n-2)-T(n))} (1 + o(1)) \right).$$

The first term in Theorem 2 corresponds to the identical permutation of the $\mathcal{T}_n$ in Theorem 1, the second term corresponds to the $\binom{n}{2}$ transpositions. $o(1)$ contains all other permutations. The proof of this estimation proceeds by estimating explicitly those permutations with a large number of fix-points and estimating uniformly the rest.

These two theorems solve completely Carnap's problem. The case m=1 was solved by Carnap himself.

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