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March 24, 1990

Mr. Seth Finkelstein

P.O. Box 1404

Cambridge, MA

Dr. N. J. A. Sloane

Dear Dr. Sloane:

I have come upon a puzzling, and possibly incorrect, reference in your work "A Handbook of Integer Sequences". I recently had occasion to research the sequence 1, 3, 13, 63, 321, 1683 ... I have found your handbook an invaluable reference for finding relevant articles in the literature, and it was very helpful that this sequence was listed. But the first reference given (MES 54 75 24) did not deal with the sequence at all. I have enclosed some photocopied pages of the journal and page cited. Since the volume and year number of the reference match, the problem cannot be a simple misprint. I thought the discrepancy sufficient to bring to your attention, and would be gratified to learn the correct citation. Thank you for your time.

Sincerely,

Seth Finkelstein

Seth Finkelstein

SF/sf

Enc.

AT&T

AT&T Bell Laboratories

600 Mountain Avenue
Murray Hill, NJ 07974-2070
201 582-3000

April 13, 1990

Mr. Seth Finkelstein
P.O. Box 1404
Cambridge, MA 02138

Returned
— addressee unknown!

Dear Mr. Finkelstein:

Thank you for your letter. You are absolutely correct. My source for that sequence is a paper by H. Bateman, "Some problems in potential theory". My photocopy is labeled *Messenger Math.*, Vol. 54 (1924) in my handwriting (which I agree is wrong), and it runs from p. 71 to p. 78. This article must be in one of the neighboring volumes of the same journal.

(I don't have access to this journal at the moment, I'm afraid.)

Thanks for pointing this out!

I enclose some recent papers of mine containing some new sequences.

With very best regards,

N. J. A. Sloane

Enc.

1181 1, 3, 12, 70, 465, 3507, 30016, 286884, 3026655, 34944085, 438263364, 5933502822, 86248951243, 1339751921865, 22148051088480, 388246725873208
POLYGONS FORMED FROM N LINES. REF CO1 2 120.

1182 1, 3, 13, 27, 52791, 482427, 124996631
ASYMPTOTIC EXPANSION OF AN INTEGRAL. REF MTAC 19 114 65.

1183 1, 3, 13, 31, 43, 67, 71, 83, 89, 107, 151, 157, 163, 191, 197, 199, 227, 283, 293, 307, 311, 347, 359, 373, 401, 409, 431, 439, 443, 467, 479, 523, 557, 563, 569, 587, 599
10 IS A QUADRATIC RESIDUE MODULO P. REF KR1 1 61.

1184 1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, 44642381823, 252055236609, 1425834724419
A SQUARE RECURRENCE. REF MES 54 75 24. SIAMR 12 277 70.

1185 1, 3, 13, 63, 326, 1761
ROOTED PLANAR MAPS. REF CJM 15 542 63.

1186 1, 3, 13, 65, 403, 2885, 23515, 214805
THE GAME OF MOUSETRAP. REF QJM 15 241 1878.

1187 1, 3, 13, 68, 399, 2530, 16965, 118668, 857956, 6369838
PLANAR TRIANGULATIONS. REF CJM 14 32 62.

1188 1, 3, 13, 70, 462, 3592, 32056, 322626, 3611890, 44491654, 597714474, 8693651092, 136059119332, 2279212812480, 40681707637888, 770631412413148
STOCHASTIC MATRICES OF INTEGERS. REF DMJ 35 659 68.

1189 1, 3, 13, 71, 465, 3539, 30637, 296967, 3184129, 37401155, 477471021, 6581134823, 97388068753, 1539794649171, 25902759280525, 461904032857319
A(N) = NA(N − 1) + (N − 3)A(N − 2). REF R1 188.

1190 1, 3, 13, 73, 501, 4051, 37633, 394353, 4596553, 58941091, 824073141, 12470162233, 202976401213, 3535017524403, 65573803186921, 1290434218669921
FORESTS OF GREATEST HEIGHT. REF RCI 194. PSPM 19 172 71.

1191 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 230283190977853
PREFERENTIAL ARRANGEMENTS. REF CAY 4 113. PLMS 22 341 1891. AMM 69 7 62. PSPM 19 172 71.

1192 1, 3, 13, 81, 721, 9153, 165313
COLORED GRAPHS. REF CJM 12 413 60 (DIVIDED BY 2). JCT 6 17 69.

1193 1, 3, 13, 83, 592, 4821, 43979, 444613, 4934720, 59661255, 780531033, 10987095719, 165586966816, 2660378564777, 45392022568023, 819716784789193
MENAGE NUMBERS. REF LU1 1 495.

1194 1, 3, 13, 87, 841, 11643
GRADED PARTIALLY ORDERED SETS. REF JCT 6 17 69.

1195 1, 3, 13, 87, 1053, 28576, 2141733, 508147108
INCIDENCE MATRICES. REF CPM 89 217 64.

1196 1, 3, 13, 146, 40422
SWITCHING NETWORKS. REF JFI 276 317 63.

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$$\begin{Vmatrix} (1\,3) & (1\,2\,3) \\ (1\,3\,4) & (1\,2\,3\,4) \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} (1\,3) & (1\,3\,4) \\ (1\,2\,3) & (1\,2\,3\,4) \end{Vmatrix} = 0,$$

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$$\begin{Vmatrix} (2\,3) & (1\,3\,4) \\ (2\,3) & (1\,3\,4) \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} (1\,3) & (2\,3) \\ (1\,3\,4) & (1\,2\,3) \end{Vmatrix} = 0.$$

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PRINTERS, CAMBRIDGE.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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ON AN EXPANSION OF $\dfrac{\pi \operatorname{sh} 2n\pi}{\operatorname{ch} 2n\pi - \cos 2\xi\pi}$ IN A SEMI-CONVERGENT SERIES.

By *N. Koshliakov.*

THE classical expansion of $\log \Gamma(t)$ into Stirling's series was generalized by Hermite and Sonin for the case when $t = x + \xi$, where $0 \leq \xi \leq 1$.

In the place of Stirling's formula

$$\log \Gamma(x) = \log \sqrt{(2\pi)} + (x - \tfrac{1}{2}) \log x - x$$
$$+ \sum_{\nu=1}^{\kappa} (-1)^{\nu-1} \frac{B_\nu}{(2\nu - 1) . 2\nu} \cdot \frac{1}{x^{2\nu-1}} + T_\kappa(x) \ldots (\mathrm{I}),$$

where $$T_\kappa(x) = \frac{(-1)^\kappa}{x^{2\kappa+1}} \cdot \frac{1}{\pi} \int_0^\infty \frac{t^{2\kappa}}{1 + \dfrac{t^2}{x^2}} \log \frac{1}{1 - e^{-2\pi t}} \, dt,$$

we have obtained the following formula

$$\log \Gamma(x + \xi) = \log \sqrt{(2\pi)} + (x + \xi - \tfrac{1}{2}) \log x - x$$
$$+ \sum_{\nu=1}^{\kappa} \frac{\phi_{2\nu}(\xi)}{(2\nu - 1) . 2\nu} \cdot \frac{1}{x^{2\nu-1}} + T_\kappa(x, \xi) \ldots (\mathrm{II}),$$

where

$$\phi_\nu(x) = x^\nu - \frac{\nu}{2} x^{\nu-1} + C_\nu^{(2)} B_1 x^{\nu-2} - C_\nu^{(4)} B_2 x^{\nu-4} + \ldots,$$

$$T_\kappa(x, \xi) = \frac{(-1)^{\kappa+1}}{x^{2\kappa-1}} \int_0^\infty \frac{2t^{2\kappa}}{x^2 + t^2} \overline{\Psi}(\xi, t) \, dt + \frac{(-1)^{\kappa+1}}{x^{2\kappa-2}} \int_0^\infty \frac{2t^{2\kappa-1}}{x^2 + t^2} \overline{X}(\xi, t) dt,$$

$$\overline{\Psi}(\xi, t) = \frac{1}{4\pi} \log (1 - 2e^{-2\pi t} \cos 2\pi\xi + e^{-4\pi t}),$$

$$\overline{X}(\xi, t) = -\frac{1}{2\pi} \operatorname{tang}^{-1} \left(\frac{\sin 2\pi\xi}{e^{2\pi t} - \cos 2\pi\xi} \right).$$

When $\xi = 0$, the formula (I) follows from (II), and when $\xi = \frac{1}{2}$, the known Gauss' expansion can be obtained.

The subject of the present note is to deduce a formula, similar to (II), for the quotient $\dfrac{\pi \operatorname{sh} 2n\pi}{\operatorname{ch} 2n\pi - \cos 2\xi\pi}$; by putting therein $\xi = 0$ we shall have an expansion of $\operatorname{cth} n\pi$ in a semi-convergent series; this expansion has been pointed out by Euler in one of his letters to N. Bernoulli.

We give here two different proofs of our formula.

1. By putting $a = n$ and $b = \nu - \xi$ and $\nu + \xi$ in the integral

$$\int_0^\infty e^{-bx} \sin ax \, dx = \frac{a}{a^2 + b^2},$$

we have

$$\sum_{\nu=1}^{n} \frac{n}{n^2 + (\nu - \xi)^2} = \int_0^\infty \frac{e^{\xi x} \sin nx}{e^x - 1} dx - \int_0^\infty \frac{e^{-(n-\xi)x} \sin nx}{e^x - 1} dx,$$

$$\sum_{\nu=1}^{n} \frac{n}{n^2 + (\nu + \xi)^2} = \int_0^\infty \frac{e^{-\xi x} \sin nx}{e^x - 1} dx - \int_0^\infty \frac{e^{-(n+\xi)x} \sin nx}{e^x - 1} dx,$$

where

$$n \sum_{\nu=1}^{n} \left\{ \frac{1}{n^2 + (\nu - \xi)^2} + \frac{1}{n^2 + (\nu + \xi)^2} \right\}$$
$$= 2 \int_0^\infty \frac{\sin nx . \operatorname{ch} \xi x}{e^x - 1} dx - 2 \int_0^\infty \frac{e^{-nx} \sin nx . \operatorname{ch} \xi x}{e^x - 1} dx.$$

On the other hand, separating the real and imaginary parts in the known Legendre's integral

$$\int_0^\infty \frac{\sin \alpha x}{e^{2\pi x} - 1} dx = \frac{1}{2} \left\{ \frac{1}{e^\alpha - 1} - \frac{1}{\alpha} + \frac{1}{2} \right\},$$

where $\alpha = a + ib, \quad a > 0, \quad b < 2\pi,$

we obtain

$$\int_0^\infty \frac{\sin ax . \operatorname{ch} \xi x}{e^x - 1} dx = \frac{\pi}{2} \frac{\operatorname{sh} 2a\pi}{\operatorname{ch} 2a\pi - \cos 2\xi\pi} - \frac{1}{2} \frac{a}{a^2 + \xi^2}, \quad \xi < 1;$$

hence

$$\frac{\pi \operatorname{sh} 2n\pi}{\operatorname{ch} 2n\pi - \cos 2\xi\pi} = \frac{n}{n^2 + \xi^2} + n \sum_{\nu=1}^{n} \left\{ \frac{1}{n^2 + (\nu - \xi)^2} + \frac{1}{n^2 + (\nu + \xi)^2} \right\}$$
$$+ 2 \int_0^\infty \frac{e^{-nx} \sin nx . \operatorname{ch} \xi x}{e^x - 1} dx.$$